JOËLLE VLASSIS

# THE BALANCE MODEL: HINDRANCE OR SUPPORT FOR THE SOLVING OF LINEAR EQUATIONS WITH ONE UNKNOWN

ABSTRACT. The use of concrete models for teaching students how to solve equations is often debated in scientific literature. This article aims to examine the balance model and to identify the issues that divide scientists. We based our reflections on the results of an empirical study and analysis of the various arguments put forward by supporters and opponents of the model. We describe learning situations that were the subject of the empirical study, which involved forty students in two 8th-grade classes. The aim was to teach the formal solving method, which involved performing the same operations on both sides of the equation using, notably, the balance model. Analysis of students' reasoning showed that the presence of negative numbers gave rise to many errors. The difficulties presented by negative numbers were reviewed, eight months later, during an interview with five students, chosen from those who took part in the experiment. Within that context, we discuss the relevance of the balance model and analyse the arguments put forward by researchers who either defend or reject its use.

## 1. INTRODUCTION

Over a period of years, a number of authors (Herscovics and Kieran, 1980; Filloy and Rojano, 1989; Radford and Grenier, 1996; Linchevski and Herscovics, 1996; Pirie and Martin, 1997) have experimented with various situations in which students learn to solve equations. Various concrete models have been put forward: the arithmagon (Pirie and Martin, 1997), the balance (Filloy and Rojano, 1989; Radford and Grenier, 1996; Linchevski and Herscovics, 1996), the geometrical model (Filloy and Rojano, 1989). These researchers arrive at conflicting results concerning the use of concrete models. What conclusions can be drawn from this?

In this article, we aim to examine the arguments put forward in these studies and to consider the role of a particular model – the balance. These reflections are based on our observations of a learning sequence that we devised, in which students learn how to solve equations with the unknown appearing in both sides of the equation, inter alia by using the balance model. Our analyses are based on classroom observations and examination of students' drafts, in which they show how they have arrived at the solution. This revealed that students experience major difficulties in solving 'non-arithmetical' equations (Filloy and Rojano, 1989) that include nega-

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tive integers. Consequently, we would like to discuss the suitability of the balance model. Eight months after the experiments, we had the opportunity to interview five of the students who had taken part in the experiments. Despite the small number of students interviewed, we believe that it is useful to refer to the results in this article.

## 2. A LEARNING SEQUENCE

### 2.1. *The context of the research*

In the French-speaking community of Belgium, solving linear equations with one unknown is part of the curriculum for 8th-grade students, who have already covered basic algebraic operations in the 7th-grade (grouping like terms, simple distributivity, multiplication of algebraic factors. . .). In actual fact, learning how to solve equations is mostly a matter of rapidly confronting the students with the formal method of 'transposition' (Kieran, 1990). This approach is justified, in that it gives students a general method that allows them to solve all kinds of linear equations with one unknown. This methodology is not very suitable for its target audience, in so far as it neglects completely any prior knowledge the students might have. The results of the TIMSS study (Henry, 1996) for the French community of Belgium say a great deal to this effect. Only 53% of students at the end of the 8th-grade are able to solve an equation such as: $10x - 15 = 5x + 20$.

### 2.2. *Methodology*

Two 8th-grade classes (40 students in total) took part in the learning activities, during the second semester of the school year. The population of the school was drawn from disadvantaged areas and standards of learning in the two classes were low, according to the teaching staff. We chose this school because the teachers were very cooperative and interested in our work. They had been working with us on a project concerning the study of algebra for two years.

In order to carry out experiments involving learning situations, the teachers taught their lessons using the materials (situation sheets, summary sheets to be completed, exercises. . .) produced by the researchers in cooperation with the team of teachers involved in the project. The complete sequence consisted of 16 sessions lasting 50 minutes (including exercises), divided into 2 phases of 8 sessions: the first phase dealt with 'arithmetical' equations and the second phase dealt with 'non-arithmetical' equations.

## 2.3. *Presentation of the sequence*

The principal objective of all the situations was the learning of the formal method for solving equations, which involved performing the same operation in the two members. Although solving the problems also required these to be set up in the form of an equation, this area was not touched upon. We believe this aspect, at least initially, should be taught separately. Authors such as Herscovics and Kieran (1980), Combier, Guillaume and Pressiat (1996) favour this approach. According to these researchers, it is important to distinguish between studying the setting up of equations and studying how equations are solved, in order to avoid an accumulation of difficulties that are inherent to each of the problem-solving phases.

The sequence consisted of a number of problems that had to be solved, according to an educational approach that makes use of 'problems solving', which we defined using precise criteria (Vlassis and Demonty, 2002). According to these criteria, we insist that the notions targeted by these situations are the most appropriate solutions. For some activities, it was necessary to set up equations. In all cases, we avoided focussing excessively upon this task of modelling, which was simplified in order to emphasise solving methods. These, after all, are the principal learning objective.

All the activities involving the students were based on the two types of linear equations with one unknown:

– *Equations with the unknown in one member* ('arithmetical' equations) and the development of intuitive methods: substitution, cover-up[1] and inverse operations. These activities had the objective of allowing the students to use 'their' arithmetical knowledge to solve the equations and to acquire the first notions relating to the concepts of equation, solution and unknown. Letters were introduced for the last situation as part of a discussion concerning the suitability of question marks[2]. In this article, we will not discuss the activities planned for this phase, as they do not relate directly to our topic.
– *Equations with the unknown on both sides of the equation* ('non-arithmetical' equations), which aimed to make students discover the formal method of solving equations, at a stage where they can see the need for such a method, and not at the outset, when they would not be able to understand its purpose.

This phase is based on three situations:

*Situation n°1: Solving a problem*

*Antoine and Sophie enter the same number into their calculators. Sophie multiplies the number by 4, and then adds 3 to this number. Antoine multiplies the number by 2 and then adds 17 to this number. They both arrive at the same result. What number did they both start with?*

*Figure 1.*

The students' knowledge was such that they were only able to find the solution by means of trial and error. One of the aims of this stage was to make them became aware of the limitations to their arithmetical methods. Another aim of this situation was also to extend the meaning of the equal sign (it is no longer a 'do something signal' (Kieran, 1981)) but becomes a symbol of equivalence placed between two expressions that must lead to the same result), and consequently, to favour an improved understanding of the concept of equation: This starts out with an intuitive understanding of an open sentence brought about by the first situations relating to 'arithmetical' equations, before its true mathematical meaning – equality of two expressions for a particular value of the unknown – is gradually understood. Following this activity, the students were set an equation with an unknown value on both sides of the equation.

*Situation n°2: The scales (extract)*

" The scales are balanced each time. In each case, calculate the value of the unknown number. "

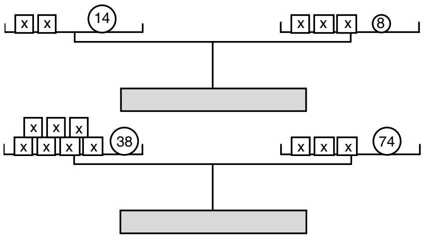



*Figure 2.*

This situation using scales served to introduce the formal method based on the properties of equality. This was introduced to the students as a tool for learning a new method, which would enable them to solve equations more quickly. The setting of equations was not introduced immediately, as it was not necessary for the problems to be solved. However, at the end of the activity, the students were invited to set up equations that corresponded to the scales. The sharing of methods used by the students (as described in point 3) made it possible to demonstrate the principles of transformation for equations. As regards notation, a method using arrows[3] on both sides of the equation to indicate the different transformations was suggested by the teachers. The students were then given exercises. These exercises mainly concerned the properties of equality based on the scales diagram, as well as the solving of equations using the balance model (requiring additions using natural numbers).

*Situation n° 3: Formalisation*

For example:

$4x + 4 + x = 2x + 13 + 6$

$13x - 24 = 8x + 76$

$-3x + 6 = 2x + 16$

*Figure 3.*

Situation n° 3 had as its aim the systematisation of the formal process, as much from the formal as from the semantic point of view. The students were required to solve 'non-arithmetical' equations without any concrete support and including negative integers.

## 3. Results

*Situation 1: Antoine and Sophie*

The students experienced many difficulties when setting up the 'arithmetical equations' necessary to begin finding the solution. Many of them needed aid. The teacher helped to establish the equations (this phase – we again emphasise – was not one of the pursued goals) and initiated the procedure by making the following suggestion: "Create an expression to express what the two children are doing". Some students used letters for the numbers, whilst others used blank spaces or question marks. The students wrote down two equations of the following type:

For Antoine: ? · 2 + 17 = ? or $x \cdot 2 + 17 = y$
For Sophie: ? · 4 + 3 = ? or $x \cdot 4 + 3 = y$

Once they had arrived at the equations, the students were able to begin finding the solution; in most cases this was by means of trial and error. The teacher then suggested condensing the two equations into a single equation:

$x \cdot 2 + 17 = x \cdot 4 + 3.$

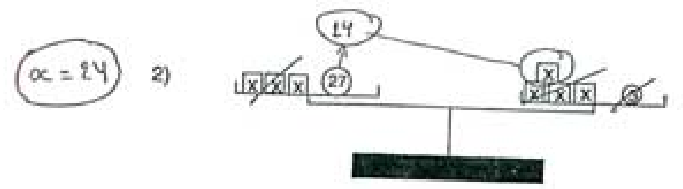
This phase was well understood by the students: "the equal sign should be placed between the two expressions, as both children arrive at the same result".

*Situation 2: The scales*

After having solved the 'Antoine and Sophie' problem, the students felt that their methods for solving problems where cumbersome and tedious. Observations made in the classes and analysis of students' drafts revealed a variety of correct methods of finding the value of weight x:

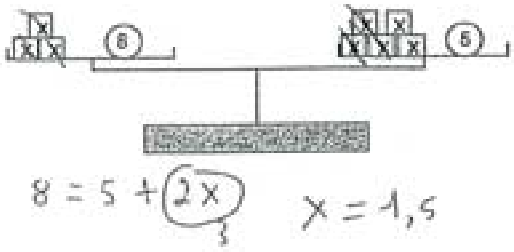
*Non-formalised methods:*
Eighteen students essentially used the drawing to find the value of x. They crossed out the same number for the weight of x on each side of the scales. When it came to the numbers, they proceeded in the same way, taking note of the new value after performing the subtraction.

For example:



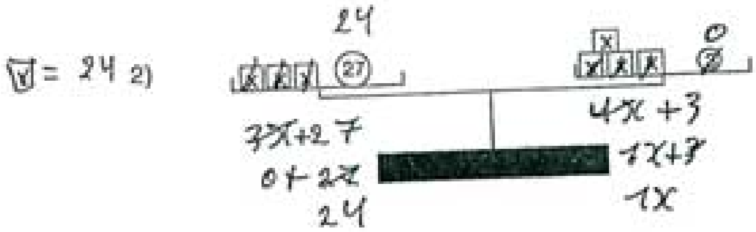

*Arithmetical methods:*
Fifteen students crossed out the x on each side of the scales and arrived at an arithmetical equation. Once they had reached this stage, they copied down the equation again and solved it using either of the two arithmetical methods (reciprocal operations, numerical recognition, or cover-up).

For example: use of cover-up




*'Algebraic' methods:*
When first looking at the scales, three students used the equivalent-equations method to solve their equation. This method is not entirely formalised, but they have mastered the principle.

For example




Following this activity, the different methods were explained and discussed in class. The teachers introduced the formal method for solving equations, based on the work of students who had suggested the formal method.

*Situation 3: formalisation*

Analysis of students' work and our field observations show that the first two activities fell into the 'zone of proximal development', as defined by Vygotsky (1962). Even if the students had to be introduced to some of the methods, they did not experience any serious difficulty in understanding their errors and integrating the new concepts.

By contrast, progressing to the 3rd situation was in no way part of this same continuum.

Our observations revealed several interesting phenomena:

1. *All of the students successfully assimilated the principle demonstrated by the scales,* i.e. performing the same operation on both sides. In general, students experienced little difficulty in attempting to solve the

equations: the majority started by removing the necessary x's, so that there were only x's in one member.

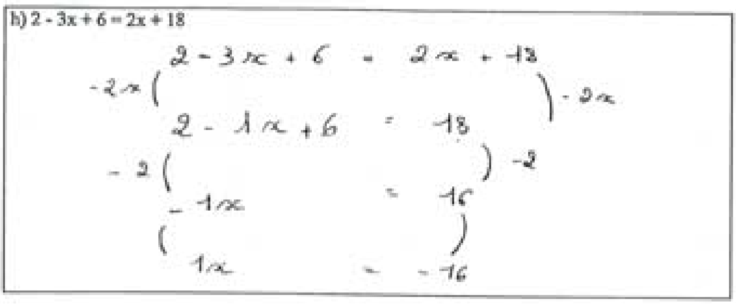
2. *However, a number of errors occurred on different levels:*

a) Some students 'divided' the two members by the coefficient of 'x', before cancelling out the independent term. For example, when solving the 1st equation of situation 3, the following transformation could be observed:
3x + 4 = 19 becomes x + 4 = 6.333

b) Students made mistakes of a syntactical nature:
One student applied, for example, an arrow '–4' to the coefficient of x and to the independent term: 4x + 4 + x therefore became x + x

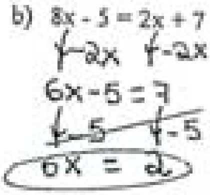
c) Many errors appeared due to the presence of negative integers in the equation, for example:

– The detachment of the minus sign (Herscovics and Linchevski, 1991)




It can be seen that this student did not consider the 'minus' sign in front of 3x as being 'attached' to 3x.

– Subtracting in order to cancel out a negative expression.
To cancel out a negative independent term (or an expression in x), some students used subtraction.
For example:




We believe that there are two possible hypotheses to explain this method. The first attributes this error to an erroneous generalisation of the balance model. The students used subtraction to cancel out an expression, as they had done in order to remove the weights from the scales. The second hypothesis refers to the first type of error mentioned above and would arise from the inability of some students to imagine the sign in front of the expression. In fact, for these students, it is not –5 that should be cancelled out but 5, the 'minus' sign in front of the 5 not having been taken into consideration.

Filloy and Rojano (1989) attribute this difficulty to the fact that the students do not manage to distance themselves from the models taught, in order to generalise the knowledge that they have acquired.

While all these errors became blurred in the exercises that the students were subsequently given, the errors involving negative integers persisted. The obstacles encountered by students in this area demanded our special attention.

## 4. Solving equations with negatives: The results from an interview

Eight months after we had worked with the classes, we interviewed five 9th-grade students from those who had taken part in the experiment (Vlassis, 2001). When we interviewed these five students, the teacher had not yet returned to the topic of solving equations. The aim of these interviews was to define more accurately the difficulties experienced by students when solving equations with negatives. The questions we asked concerned not only 'non-arithmetical' equations, but also 'arithmetical' equations. We will summarise briefly the results that are relevant to our topic:

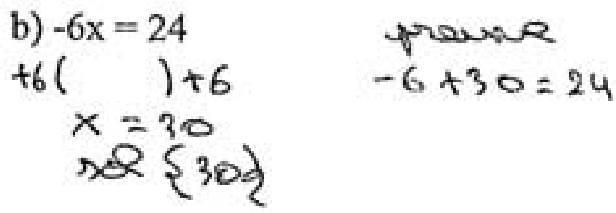
a) *Arithmetical equations*
The following are examples of each type of equation: i) $12 - x = 7$; ii) $4 - x = 5$; iii) $-4 - x = 10$; iv) $-x = 7$. All the students interviewed experienced difficulties when solving equations ii, iii and iv. These difficulties stemmed mostly from the fact that the students were attempting to solve them in an arithmetical way and were incapable of making sense of this type of equation. The equation $-x = 7$ was the most difficult equation to solve.

b) *Non-arithmetical equations*
The students immediately started to solve the equations by returning to the balance model and using the same operation on both sides.

However, the introduction of negatives brought with it two major types of errors:

1. The ‘detachment of the minus sign’ (Herscovics and Linchevski, 1991).
2. The inability to isolate x when it is preceded by a negative coefficient:

   For example:




These interviews involved a small number of students and it would be wrong to generalise these results to include all of the 40 students that took part in the experiments. However, the following tendencies can be identified and more attention will be devoted to these at a later stage:

- The method of solving equations based on the properties of equality seems to have been assimilated. Let us not forget that the students had not yet returned to this topic at the start of the year and that they remembered very quickly the transformations to be carried out.
- The negatives place the equation (‘arithmetical’ or ‘non-arithmetical’) on an abstract level. It is no longer possible to refer back to a concrete model or to arithmetic. The “didactical cut” does not seem to depend upon the structure of the equation (unknown on both sides of the equation), but upon the degree to which the equation has been made abstract by the negatives. Arithmetical equations with negatives therefore also represent an obstacle for those students who are unable to give them a concrete meaning.
- None of the five students interviewed subtracted a negative expression in order to cancel it out. This result must of course be qualified, as only a small number of students were interviewed.

## 5. Beyond the 'Arithmetical' and 'Non-Arithmetical' Equations Dichotomy

The results obtained reveal that it is necessary to return to the distinction that Filloy and Rojano (1989) make between 'arithmetical' and 'non-arithmetical' equations. This distinction is based on the 'didactical cut' that separates equations with the unknown in one member from equations with the unknown on both sides. This 'didactical cut' comes into play whenever students are called upon to operate on the unknown, when dealing with "non-arithmetical' equations. The results that we obtained both from our experiments during the sequence and from the interviews prompt us to qualify this categorisation.

*Arithmetical equations:*

According to Filloy and Rojano (1989), this category includes equations with the unknown in one member, where the equal sign retains its arithmetical significance. The label 'arithmetical equations', applies equally to equations such as x + 7 = 15 and -x = 7 or even 6x + 5 – 8x = 27. It can be seen that the last two types of equations cannot be solved entirely by means of arithmetic. To decode the equation -x = 7 requires either approaching it in terms of –1 . x = 7 or by means of opposites, whereas the equation 6x + 5 – 8x = 27 requires operating on the unknown. The only thing this equation has in common with an equation such as x + 7 = 15 is the fact that the unknown is only to be found in one member and that the equal sign can still be interpreted arithmetically. We therefore would like to retain the 'arithmetical equations' label to designate equations where the unknown features in one member, whilst making the following qualifications:

- *Concrete arithmetical equations:* This category includes arithmetical equations that consist only of natural numbers and only include a single occurence of the unknown.
- *Abstract arithmetical equations*: This category includes equations with the unknown in one member, but which require certain algebraic manipulations, for example, because of the presence of negative integers or several occurences of the unknown.

*Non-arithmetical equations*

This category includes all equations with the unknown on both sides of the equation. On the basis of our results, it seems, however, that a distinction must be made between equations that are 'based on a model' and those that are 'detached from a model':

- *Equations that are 'based on a model' or 'pre-algebraic' equations:* These equations are derived from models and frequently involve additions of naturals. They may, however, include subtractions, as it is the case with the situations set in a geometrical context by Filloy and Rojano (1989), or those proposed by Radford and Grenier (1996). We qualify these equations as being 'pre-algebraic', in so far as they are based on a structure suggested by a concrete model, whilst necessitating an algebraic understanding of the equal sign.
- *Equations that are 'detached from models' or 'algebraic equations':* The mathematical objects (letter, numbers,...) that are suggested by these equations no longer refer back to a concrete model. The manipulations only have meaning in an algebraic context.

It is important to understand this clarification of the nature of equations, as the categories suggested by Filloy and Rojano (1989) could have us believe that 'arithmetical' equations are more accessible for students than 'non-arithmetical' equations, whereas the difficulty lies not in the structure of equations, but in their degree of abstraction. The question should therefore be: "Can the equation be solved by referring to the concrete or not?". We believe that this analytical axis enables us to predict more effectively the potential difficulties for students.

## 6. THE USEFULNESS OF MODELS

There are two theoretical tendencies with conflicting views of the usefulness of models:
*The opponents*: Filloy and Rojano (1989) observe that the models (geometrical and balance) do not allow students to deal with the unknown value. Pirie and Martin (1997) believe that the balance model makes no sense to modern students, as contemporary scales are no longer based on the principle of the two sides balancing. Moreover, these authors argue that using this model gives rise to errors that are caused by the model itself (such as removing a negative integer to cancel it out).
*The defenders*: On the other hand, Brown, Eade and Wilson (1999) are of the opinion that "for many students and many teachers proficiency in specific concretisations forms the backbone and principal motivation of activity within the classroom." (p.68). The results of Radford and Grenier (1996) show that "the idea of the scales facilitates the use of the rule of elimination of like terms (rule of the al-muqabala)" (p.264). Moreover, according to Linchevski and Williams (1996): "people attempting to solve mathematical problems often make use of several models in the process of

finding the solution. Different parts of the problem may lend themselves to the use of different representations, including the combination of concrete thinking with abstract formal reasoning" (p.266).

However, all of these authors agree that the models are not effective once negative integers appear on the scene. In the case of our experiment, we believed that this transition would be less difficult, as the students had been studying algebra for one year – which was not the case for the studies described in this article – but the results show that the gulf was equally difficult to cross.

Filloy and Sutherland (1996) speak of the 'separation' of the meaning introduced by the model for solving abstract situations. What does this "separation" of the model consist of, in the scope of our learning sequence?

### 6.1. *The detachment of the model*

Analysis of our students' drafts shows that the presence of negative integers, because of their abstract nature, necessitates an algebraic understanding of letters, which is in conflict with that introduced by the balance model. We will analyse the way in which negatives and the development of the interpretation of letters cause difficulties for students.

*Negative integers*

For Gallardo (2001), "the extension of the numerical domain from natural numbers to integers becomes a crucial element for achieving competence in the solution of problems or equations" (p. 128). Historically, we must not forget that this development took place gradually and that it was only in the $19^{th}$ century that this extension took place (Glaeser, 1981). Our interviews show that the difficulties experienced by students stem from the fact that students have not yet reached this stage. In fact, the errors observed while the students were attempting to solve equations with negatives suggest a formal understanding of these numbers, as in the example mentioned, in which a student failed to progress beyond the $-5x = 10$ to $x = -2$ stage. As far as the concept of negative solutions is concerned, this has given rise to 'inhibitory mechanisms' when solving arithmetical equations, as already highlighted by Gallardo and Rojano (1994). Strangely, this problem did not occur in the case of negative solutions to 'algebraic' equations. We are making the assumption that in the case of these equations, it was not necessary to give a meaning to this solution, whilst for the first equations, the students – who for the most part used the substitution (for $4 - x = 5$: what number produces 5 when it is subtracted from 4?) – had to give a meaning to this solution to solve the equation.

*The interpretation of letters*
Whilst the equations implied in situations n°2 and n°3 present the same structure: ax±b=cx±d, the strategies used to solve them (which were apparently similar) imply a different understanding of letters. In fact, in equations that are 'based on the model', students are prompted to imagine the letter as a quantified object, i.e. a weight of an unknown numerical value. This interpretation needs to be related to the semi-concrete notion mentioned in the work of Radford and Grenier (1996), where the use of the letter 'e' still presented a very strong link with the object represented (the envelopes). Whenever negatives appear in equations that are 'detached from the models', it no longer serves any purpose to consider the letter as a weight, even with a numerical value. A leap forward towards a true algebraic notion of letters is necessary. How can this notion be defined? What distinguishes it from those suggested in arithmetic?

In arithmetic, letters do not exist in their own right, and exist only by virtue of the concrete meaning that the student gives them directly. They only have a meaning for students as 'objects' (Küchemann, 1981) or numbers. In this last case, the student gives the letter a value either by direct recognition or by trial and error. Alternatively, its value will be the result of a series of inverse operations.

An algebraic interpretation implies that the student is able to refrain from immediately attributing a concrete meaning to the letter (number or object) and to interpret it as an unknown number, the value of which is not significant for the time being. In order to define this notion, Küchemann (1981) speaks in terms of a 'specific unknown' that characterises students who see the letter as a specific yet unknown number and who are able to apply operations directly to this letter. With the unknown letter, the meaning of the operations is altered and they take on another status. Slavitt (1999) states that the ability to manipulate the unknown "not only does involve the use of the operation without immediate concrete referents, but it also illustrates an ability to use the operation without specific objects being signified by the inputs' referents" (p. 257) . . . This ability "requires acts of generalization and places the primary focus on the operation itself". This is very different from arithmetic, where the emphasis is placed mainly on the components: *les nombres sans signe* (Vergnaud, 1989) or letters with an arithmetical meaning (question marks or spaces in open sentences).

We believe there is an intermediate stage between arithmetical interpretation and algebraic interpretation of the letter, which is the stage suggested in 'pre-algebraic' equations. This level would characterise those students, who are capable of performing operations on the letter, whilst retaining a meaning linked to the model. This would cause problems for them when they came to generalise their methods in abstract situations, caused particularly by the presence of negatives.

Following this analysis, it is quite reasonable to question the usefulness of the balance model. It appears to be less effective. Furthermore, might it not risk creating an obstacle when the time comes to move on to abstract manipulations?

### 6.2. *Should the balance model be rejected?*

We are in favour of the use of models, and particular the balance model – for the following reasons:

- Our results show that the balance model is especially suited to the study of how to solve equations. In fact, the isomorphism between the object itself and the mathematical notions implied allows students to form a mental image of the operations that they have to apply. They are able to reactivate this self-evident image at any moment.
  Our research has shown that the solving of algebraic equations implies that students must have assimilated at least three basic skills: 1) the principle of transformations in equivalent equations (performing the same operation on both sides), 2) having extended their numerical range with negative integers and 3) understanding the letter as an unknown. An analysis of the students' drafts shows the effectiveness of this tool in conveying the principles of transformations. Where scales are used, they provide a mental picture of the manipulations to be carried out and the associated concepts (the meaning of equality and the expressions, the properties of equality), but they do not help students in the two other situations. We are therefore of the opinion that the apparent inefficiency of the models is caused rather by a flawed understanding of their relevance.
- Basing the study of how to solve 'algebraic' equations on a model, such as that of the scales provides the students with the principles they need to perform transformations, summarised in a single and self-evident image. We talk of 'operative' mental images, i.e. images that contain the same principles as they introduce. This tool is more efficient in terms of the demands on memory than a description (however meaningful) of the operations to be carried out. Eight months later and without any prior warning, our students had no difficulty in reactivating this image by applying the principles correctly.
- Even critics of the balance model (such as Pirie and Martin, 1997) emphasise its effectiveness in explaining the meaning of the equal sign and the image 'at one moment' of the equation that the model provides. On the other hand, students assimilate only very gradu-

ally the concepts implied by the properties of equality and by the acquisition of a 'structural' view of expressions (Sfard, 1991). The old-fashioned image of the scales, as presented by the model, has not in any way created an obstacle in the eyes of the students. Of course, it was necessary to explain this to the students, but they had clearly comprehended that it was not the physical object itself that was important, but the principle of balance that it demonstrated.

- Pirie and Martin (1997) show that models such as the balance can give rise to errors linked to the model itself, such as subtracting a negative number in order to cancel it out. The results of our experiments show that these errors indeed appeared during the first teaching session. The same errors become more blurred in the students' drafts and did not recur in the case of the five students interviewed.
- According to Filloy and Rojano (1989), in the course of their transformations, students lose skills that they have already acquired. For example, students appeared unable to identify arithmetical equations, and therefore were unable to solve them. This problem did not occur in our experiment. On the contrary, as soon as some students reached an arithmetical equation, in situation 2 and even 3, they concluded their solution by using an arithmetical method. The importance attached to the solving of arithmetical equations in our sequence has certainly helped students to retain their arithmetical skills.
- All of these reflections reveal that efficient use of the models demands a thorough analysis of the obstacle that they are supposed to overcome and also the obstacles that are involved in the learning objectives. By combining both points of view, one can avoid attributing any objectives to the models for which they were not intended.

Finally, we would like to emphasise that the process of abstraction is complex and that it consists of several dimensions. Different approaches will be necessary if this objective is to be achieved. Concrete models such as the scales have a part to play in this process, a part that is not universal, but which nonetheless has its place in the mathematical training of students.

## CONCLUSIONS

Our observations show that the balance model can certainly help students to learn the formal method of applying the same operation in the two members. Its essential interest consists not only of giving a concrete meaning to these manipulations, but also in providing students with an 'operative' mental image that contains the principles to be applied. Observation of our students shows that this effect is long lasting: eight months after the experiment, the students were still using the principle correctly by remembering the image of the scales. This was an important achievement. Research literature has emphasised sufficiently the difficulties at this level. This image can also be useful for students in older grades, as a safety net to which they can turn, if ever they lose their way whilst performing complex procedures. However, the solving of equations that are 'detached from a model' implies that other obstacles have to be overcome that are linked to the process of abstraction itself (unknown, negative numbers, . . .) and for which the balance model is not intended. Other activities are necessary that will allow students to distance themselves from the scales, while retaining the principles of the transformations that they introduce.

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## NOTES

1. This method is an extension of the substitution method. The cover-up method involves considering as the first unknown the expression that contains this unknown. This causes students to solve an equation such as 2r+7=35, by considering the expression 2r initially as the unknown. The equation **2r**+7=35 becomes **x**+7=35. The value of 2r is therefore 28; the value of r is therefore 14.
2. The discussion concerned the resolution of an equation with several occurrences of the unknown (indicated by a question mark) in the first member: Did the question marks replace the same number or different numbers? Using the same letter had the advantage of clarifying matters to this regard.
3. The arrows must be considered as belonging to the concrete *elementary iconic symbolic language* (EISL), as defined by Radford (2001). This language is characterised by a less rigid syntax than formal syntax and has the advantage of providing a friendly environment for non-expert users of symbolic languages.

## REFERENCES

Booth, L. and Cook, J.: 1988, 'Children's difficulties in beginning algebra', in A. Coxford and A. Shulte (eds.), *The Ideas of Algebra*, K-12. National Council of Teachers of Mathematics, pp. 20–32.

Brown, T., Eade, F. and Wilson, D.: 1999, 'Semantic innovation: Arithmetical and algebraic metaphors within narratives of learning', *Educational Studies in Mathematics* 40, 53–70.

Colomb, J.: 1995, *Calcul littéral. Savoir des élèves de collège*, Institut National de Recherche pédagogique. Document et travaux de recherche en éducation, Paris.

Combier, G., Guillaume, J.-C. and Pressiat, A.: 1996, *Les débuts de l'algèbre au collège. Au pied de la lettre,* Institut National de Recherche Pédagogique, Didactiques des disciplines, Paris.

Filloy, E. and Rojano, T.: 1989, 'Solving equations: The transition from arithmetic to algebra', *For the learning of the mathematics* 9(2), 19–25.

Filloy, E. and Sutherland, R.: 1996, 'Designing curricula for teaching and learning algebra', in A. Bishop (ed.), *International Handbook of Mathematics Education*, vol. 1, Kluwer Academic Publishers, Dordrecht, pp. 139–160.

Gallardo, A.: 2001, 'Historical-epistemological analysis in mathematics education: two works in didactics of algebra', in R. Sutherland, T. Rojano, A. Bell and R. Lins (eds.), *Perspectives on School Algebra*, Kluwer Academic Publishers, Dordrecht, pp. 121–139.

Gallardo, A. and Rojano, T.: 1994, 'School Algebra. Syntactic difficulties in the operativity', *Proceedings of the XVI International Group for the Psychology of Mathematics Education, North American Chapter*, vol. 1, USA, Louisiana State University, pp. 265–272.

Glaeser, G.: 1981, 'Epistémologie des nombres relatifs', *Recherche en didactique des mathématiques* 2(3), 303–346.

Henry, G.: 1996, *Troisième étude internationale en Mathématiques et en Sciences. Epreuves de rendement-Mathématiques*, Rapport technique. Service de Développement et d'Evaluation des Programmes de Formation de l'ULg.

Herscovics, N. and Kieran, C.: 1980, 'Constructing meaning for the concept of equation', *The Mathematics Teacher* 73(8), 572–580.

Herscovics, N. and Linchevski, L.: 1991, 'Pre-algebraic thinking: Range of equations and informal solution processes used by seventh graders prior to any instruction', *Proceedings of the XV Annual Meeting of PME*, vol. 3, Assissi, Italy, pp. 173–180.

Kieran, C.: 1981, 'Concepts associated with equality symbol', *Educational Studies in Mathematics* 12, 317–326.

Kieran, C.: 1990, 'Cognitive processes involved in learning school algebra', in P. Nescher et J. Kilpatrick (eds.), *Mathematics and Cognition*, Cambridge University Press, Cambridge, pp. 96–102.

Kieran, C.: 1992, 'The learning and the teaching of scool algebra', in D. Grouws (ed.), *Handbook of research on Mathematics Teaching and Learning*, Mac Millan, New York, pp. 390–419.

Küchemann, D.E.: 1981, Chapitre 8: 'Algebra', in K.M. Hart (ed.), *Children's Understanding of Mathematics,* The CSMS Mathematics Team, London, pp. 11–16.

Linchevski, L. and Herscovics, N.: 1996, 'Crossing the cognitive gap between arithmetic and algebra: Operation on the unknown in the context of equation', *Educational Studies in Mathematics* 30, 39–65.

Linchevski, L. and Williams, J.S.: 1996, 'Situated intuition, concrete manipulations and the construction of mathematical concepts: The case of integers', *Proceedings of the 20th conference of the International Group for the Psychology of Mathematics Education*, University of Valencia, Valencia, Vol. 3, pp. 265–272.

Pirie, S. and Martin L: 1997, 'The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations', *Educational Studies in Mathematics* 22, 1–36.
Radford, L.: 2001, *'Algebra as tekhnè. Artefacts, Symbols and Equations in the Classroom*', Pre-print n°2. Université Laurentienne, Ecole des sciences de l'éducation, Ontario, Canada.
Radford, L. and Grenier, M.: 1996, 'Les apprentissages mathématiques en situation', *Revue des Sciences de l'éducation* XXII, 2, 253–276.
Sfard, A.: 1991, 'On the dual nature of mathematical conceptions', *Educational Studies in Mathematics* 22, 1–36.
Sfard, A. and Linchevski, L.: 1994, 'The gains and the pitfalls of reification: The case of algebra', *Educational Studies in Mathematics* 26(2–3), 191–228.
Slavitt, D.: 1999, 'The role of operation sense in transitions from arithmetic to algebraic thought', *Educational Studies in Mathematics* 37(3), 251–274.
Vergnaud, G.: 1989, 'L'obstacle des nombres négatifs et l'introduction à l'algèbre', in N. Bednarz & C. Garnier (eds.) *Construction des savoirs, Obstacles et conflits*. Editions Agence d'ARC Inc, Ottawa, pp. 76–83.
Vlassis, J.: 2001, 'Solving equations with negatives or crossing the formalizing gap', *Proceedings of the 25th conference of the International Group for the Psychology of Mathematics Education*, July 11–17, 2001, Utrecht, The Netherlands, Volume 3, pp. 375–382.
Vlassis, J. and Demonty, I.: 2000, *Épreuve d'algèbre en 2e année du secondaire: Résultats de population*, Rapport de recherche, Liège: Service de Pédagogie expérimentale de l'Université.
Vlassis, J. and Demonty, I.: 2002, *L'algèbre par des situations-problèmes*, De Boek, Bruxelles.
Vygotsky, L.: 1962, *Thought and Language* MIT Press, Cambridge.

*Licenciée en Sciences de l'Education,*
*Chercheur au Service de Pédagogie expérimentale,*
*UNIVERSITE DE LIEGE,*
*Sart Tilman – Bât. B32,*
*4000 LIEGE,*
*Belgique*

*Telephone 00-32-4-366 20 58-75, Fax 00-32-4-366 28 55,*
*E-mail: joelle.vlassis@ulg.ac.be*