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CHAPTER NINE

USING MANIPULATIVES FOR TEACHING EQUATION CONCEPTS IN LANGUAGING-BASED CLASSROOMS

Daranee Lehtonen and Jorma Joutsenlahti

Introduction

Conceptual understanding is one of the most important proficiencies in mathematics. It enhances students' learning fluency and retention, facilitates their learning of new concepts, helps them to avoid errors, and promotes self-discovery (NRC 2001, 116–20). Moreover, it has been acknowledged that students' low performance in mathematics can result from an inadequate understanding of mathematical concepts (NRC 2001, 17–18; Ojose and Sexton 2009, 4). Nevertheless, school mathematics has typically emphasised algorithmic skills (Attorps 2006, 1; NRC 2001, 4). Recently, several countries have reformed their mathematics curricula in favour of conceptual understanding, instead of relying entirely on algorithms (e.g. Australian Curriculum, Assessment and Reporting Authority 2015; Common Core State Standards Initiative 2010; Finnish National Board of Education 2015).

Based on the work of Piaget, Bruner, and Montessori, educators and researchers have advocated the use of manipulative materials as hands-on learning tools for mathematical concepts understanding (McNeil and Jarvin 2007, 310; Uttal et al. 2013, 2). Previous studies have demonstrated that manipulatives assist children in developing their understanding of abstract mathematical concepts through multimodality and experiential learning (Puchner et al. 2010, 314; Uttal et al. 2013, 2). On the other hand, there is also a considerable amount of research that has demonstrated that manipulatives have no benefits to learners, and can sometimes even obstruct their learning (Martin, Svihla, and Smith 2012, 1–2; McNeil and Jarvin 2007, 312; Uttal et al. 2013, 2). The fact that the benefits of manipulatives are debatable has therefore caused uncertainties when it comes to applying them in practice.

To establish whether it is worth utilising manipulatives in mathematics classrooms, we compared classes using manipulatives to classes that did not. One-variable linear equations in third- to sixth-grade classes were used as a case study for our investigation because this important concept in algebra has usually been taught merely in terms of rules and procedures, rather than focusing on the concepts contributing to those rules (Magruder 2012, 13; NRC 2001, 259). This chapter attempts to use the studied context to resolve the disagreement over the use of manipulatives in practice. First, it reviews some of the proposed reasons that manipulatives may not be beneficial, and could even be damaging to children's learning and achievement. It also reviews current suggestions for benefiting from manipulatives and relevant models. Second, it presents the context, methods, and results of our study. Finally, it discusses the observed benefits of manipulatives in the studied context, and then proposes evidence-based implications for research, practice, and policy.

Literature review

Research into the effectiveness of manipulatives has yielded varying results, suggesting that their use alone may not automatically facilitate learning within mathematics classes. While there have been many explanations as to why earlier research concluded that the use of manipulatives is ineffective, some of these explanations have actually reached the opposite conclusion. However, several of the proposed explanations do signal the same conclusion: that is, there are potential advantages of using manipulatives, but that they do have to be used appropriately and effectively. Two recommendations regarding how to benefit from manipulatives can be drawn from previous studies. First, manipulatives should be used for fostering children's conceptual understanding rather than for acquiring procedural fluency. Second, while interacting with manipulatives, children need to make a connection between different representations constructed through the manipulatives and mathematical symbols of the same concept. (e.g. McNeil and Jarvin 2007; NRC 2001; Uttal et al. 2013.)

According to the recommendations from previous studies, using manipulatives to facilitate linking various representations of mathematical concepts together can contribute to students' conceptual understanding. To date, various translation models of multiple representations in learning mathematical concepts have been recommended (e.g. Goldin and Shteingold 2001; Joutsenlahti and Kulju 2010; Lesh, Landau, and Hamilton 1983). Besides proposing different representations of mathematical concepts, they have emphasised that "representational fluency"—which has been defined as (a) the ability to represent to-be-learnt mathematical concepts in various forms, and (b) the ability to bridge these representations—plays an important role in facilitating children's understanding of mathematical concepts. Several other studies have supported this understanding (e.g. NRC 2001; Suh and Moyer 2007; Teck 2013).

Our research employed "languaging mathematics", one of the translation models proposed by Joutsenlahti and Rättyä (2015), to enhance students' representational fluency while interacting with manipulatives. In this chapter, we refer to languaging mathematics as "languaging". The term languaging was previously introduced to didactic mathematics and second language learning in relation to verbal communication (see Bauersfeld 1995; Swain 2006). However, Joutsenlahti and Rättyä's (2015) concept of languaging goes further. They have defined languaging as an approach where a student expresses their own mathematical thinking by using one or more of the following four types of language: natural (verbal and written), pictorial, mathematical symbolic, or tactile language. Tactile language has been added to the current model so as to take account of mathematical thinking occurring when interacting with hands-on materials (i.e. manipulatives). Languaging-based instruction has been studied at different educational levels (Joutsenlahti and Rättyä 2015, 51–53). Based on these studies, and those of other researchers (e.g. Bauersfeld 1995; Suh and Moyer 2007; Teck 2013), it has been demonstrated that languaging plays a crucial role in mathematics classrooms in three aspects: the development of students' conceptual understanding, co-operative learning, and the assessment of students' mathematical thinking and learning.

Recently, the new Finnish National Core Curriculum for Basic Education (first to ninth grades) has emphasised mathematical concepts understanding as one of the most important mathematical proficiencies the curriculum aims to develop among students. Concrete and experiential teaching and learning have been underlined as a key instructional method. Additionally, languaging-based class activities have been included in the curriculum. Students are encouraged to develop their mathematical thinking and present it to their classmates and teachers through concrete tools, spoken and written language, and drawings (FNBE 2015, 128, 234–35, 374).

Context and methods

To be able to decide whether the use of manipulatives should be adopted into practice, we used one-variable linear equations in third- to sixth-grade classes as the lens through which the benefits of manipulatives were investigated. We conducted cross-sectional case studies utilising a concurrent triangulation approach of mixed methods as a strategy of inquiry. Qualitative and quantitative data were collected from teachers and students and then integrated for data analysis in order to holistically combine the research findings.

Participants

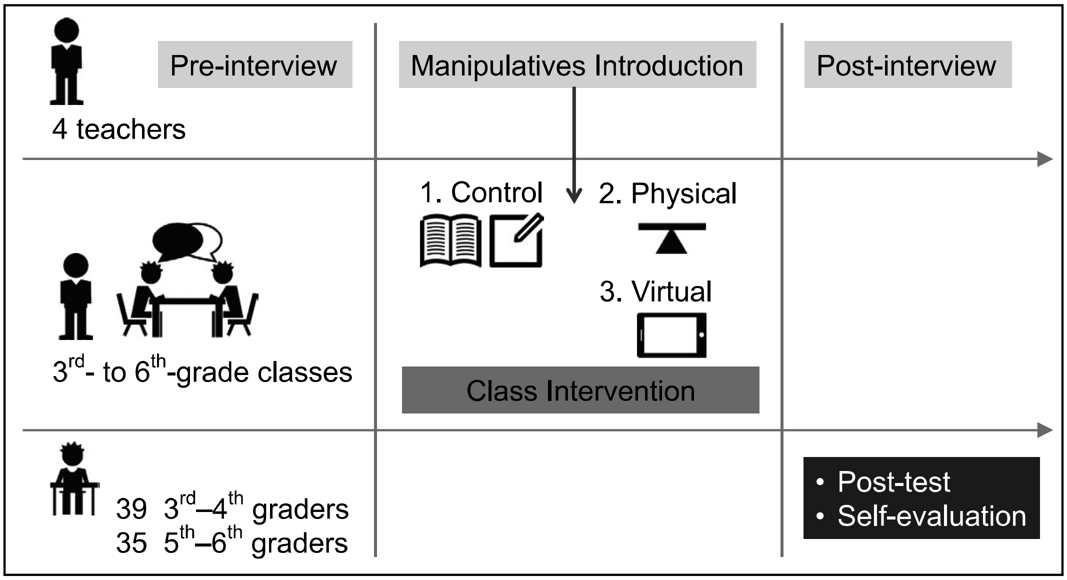
The study was conducted in one third-grade, one fourth-grade, one fifth-grade, and one sixth-grade class in a typical middle-size lower comprehensive school in southern Finland. This particular setting was selected as a case study for two reasons. First, schools and classes in Finland are homogeneous in terms of the students' socioeconomic background and performance (FNBE 2012, 2, 14; OECD 2013, 5–6). Moreover, all permanently employed class teachers in Finnish schools are required to have the same qualifications, including a master's degree and continuing professional development (OECD 2013, 10–11). Consequently, the homogeneity of Finnish comprehensive schools and class teachers made it possible for us to conduct the research in any Finnish school. Second, with limited resources and time, we expected to achieve the most fruitful results by studying third- to sixth-grade classrooms, in which the use of manipulatives has usually declined (Marshall and Swan 2008, 344).

Four class teachers (teaching experience 6–21 years) and 74 students (ages 9–12, N_{3rd}=23, N_{4th}=16, N_{5th}=14, N_{6th}=21) from the school participated in the study. Teachers' pre-interviews revealed that none of them had ever used the manipulatives intended for this study. Due to the mathematics contents included in the previous and new National Core Curriculum (FNBE 2015, 236, 375), all teachers had limited experience in teaching equations. Moreover, the students had low prior experience and knowledge of the mathematical content used in this study. Third- and fourth-grade students had not received any formal instruction in equations, while fifth- and sixth-grade students had received some instruction in solving one-variable linear equations with trial-and-error substitution of values and reasoning for the unknown. It could therefore be claimed that the homogeneity of the participants helped ensure validity and credibility of conclusions to be drawn from the research results.

Procedures

Four separate studies utilising identical research methods and procedures were conducted in the participants' classrooms during regular school hours. The studies were grouped into two grade bands (third- to fourth-grade and fifth- to sixth-grade) according to their similarity of instructional and post-test materials. Each study consisted of the following: 1) teachers' pre- and post-interviews; 2) class intervention, including one control group (a languaging-based classroom without manipulatives) and two treatment groups (a languaging-based classroom with a physical or virtual manipulative); and 3) students' post-test and self-evaluation (Figure 9-1).

Figure 9-1. Mixed-method research design



Teachers' interviews. The teachers participated in face-to-face semi-structured interviews before the treatment groups' lesson to illuminate any prior conceptions and experiences in teaching equations and utilising manipulatives which might affect the study. After all the class interventions, another interview was held to learn about their experiences, perceptions, and opinions about teaching and learning during the interventions.

Class interventions. Based on the students' prior mathematics performance during the academic year, each class teacher categorised them, within-class, into low, medium, and high attaining. They then assigned students from each category randomly, to either a control group or one of the two treatment groups. This was to ensure similarities between the instructional groups; that is, an equal number of students from each attaining level in all groups ($N_{\mathbf{control}}$=25, $N_{\mathbf{physical}}$=25, $N_{\mathbf{virtual}}$=24). The same teacher taught the control and the treatment groups the same content for one 45–minute lesson. Before the study, each class teacher received instructional materials—a teacher guide and a student worksheet that were specially designed for the study to ensure conformity between the four studies. To avoid the influence of manipulatives on the control group's session, the manipulatives intended for the treatment groups' lesson were introduced to each teacher after the control group's lesson.

The lessons in all groups were almost identical. They each: 1) learnt the concepts of equivalence and unknown; 2) developed representational fluency through languaging—that is, translating and making connections between various representations (verbal and written, pictorial, and mathematical symbolic) of equations; 3) solved equations; and 4) checked the solutions. The only difference was that the treatment groups utilised the provided manipulatives (tactile language) to accompany their lesson. According to our literature review, one drawback of previous research on physical and virtual manipulatives for equations is that the manipulatives used in the research differed from each other in several ways (e.g. Magruder 2012; Suh and Moyer 2007). Consequently, the various dissimilarities of the manipulatives make it difficult to compare the research results in terms of representational difference. Thus, this study utilised a physical and a virtual manipulative that shared the same concept and a similar operation for the treatment groups in order to minimise the effect of their other differing attributes on the research results. During the lesson, one of the treatment groups utilised Hands-On Equations® consisting of a balance scale, number cubes representing constants, and pawns representing variables, while another group utilised a virtual version of physical Hands-On Equations® for the iPad, Hands-On Equations 1 applet (Figure 9-2).

Figure 9-2. Hands-On Equations® and Hands-On Equations 1 applet

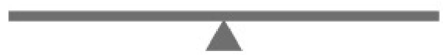
The instructional materials were divided into two sets, one for the third- and fourth-grade studies and another one for the fifth- and sixth-grade studies. There were only two differences between the two sets. First, the third- and fourth-grade lessons addressed equations with a pictorial unknown and solving equations by trial-and-error substituting values and reasoning for the unknown, whereas the fifth- and sixth-grade lessons addressed equations with a letter as an unknown and equations solving by performing the same operation on both sides of the equation. Second, the number values used in the fifth- and sixth-grade lessons required more arithmetic skills than the ones used in the third- and fourth-grade lessons.

Students' post-tests and self-evaluations. After the class interventions, all students completed the same 45–minute post-test with no access to the manipulatives. The test was administered to determine the relative difference in students' learning achievement across instructional conditions. Two post-tests (one for all third and fourth graders and another one for all fifth and sixth graders) were designed in a similar way to the class intervention worksheets. Each post-test contained six open-response test items requiring students to: 1) translate six equations presented through different representations (written, pictorial, or mathematical symbolic) into two other representations; 2) solve the value of unknowns; and 3) algebraically check their solutions (Figure 9-3). Furthermore, the third and fourth graders had to explain the strategies they used to find the unknown's value, while the fifth and sixth graders had to write down their steps of solving equations. After completing the post-test, students evaluated their learning experiences and achievement.

Figure 9-3. Three types of fifth- and sixth-grade post-test items

1. Visualize the equation below on the balance scale. Explain the equation in your own words. Solve and check the equation. Provide the steps of your solution.

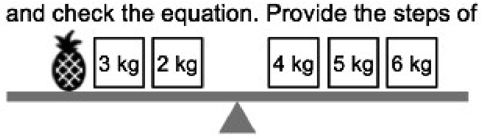
   $x + 7 = 4 \bullet 8$



The equation means that
Solution:

x =
Check:

2. Explain the picture below in your own words. Represent the picture with mathematical equation. Solve and check the equation. Provide the steps of your solution.



Equation:
Solution:

One pineapple weighs kg
Check:

3. Visualize the word problem below on the balance scale. Represent the word problem with mathematical equation. Solve and check the equation. Provide the steps of your solution.

   When a mother is weighing ingredients for her cake, she notices that three eggs weigh as much as 20 g of butter and 25 g of flour together.

Equation:
Solution:

One egg weighs g
Check:

Results and discussion

Quantitative data from the post-tests and self-evaluations of students in both grade bands were used to statistically determine whether languaging-based learning with physical or virtual manipulatives enhanced the students' understanding of equation concepts compared with the control groups. Additionally, qualitative data from the teachers' pre- and post- interviews, along with the classroom intervention observations, were concurrently utilised to develop empirical understanding of the research results. Subsequently, all the data was integrated and then interpreted to cross-validate the findings. To address the question of whether manipulatives should be adopted into practice, we next provide and discuss our findings according to our research methods (i.e. students' post-tests and self-evaluations, teachers' pre- and post-interviews, and class intervention observations) before finally turning to our convergent research results.

Students' post-tests

To determine the impact of each instructional condition on students' learning, we examined students' post-tests across three instructional conditions in both grade bands ($N_{3rd\text{-}4th}$=39, and $N_{5th\text{-}6th}$=35). Overall, both manipulative groups of both grade bands out-performed the control groups on the post-test (Figures 9–4 and 9–5). The third- and fourth-grade physical manipulative groups had the highest post-test average scores (Mean=17.7 out of 24, SD=4.0), followed by the virtual manipulative (Mean=15.9, SD=5.5) and the control groups (Mean=13.6, SD=5.0). Similarly, the fifth- and sixth-grade physical manipulative groups performed better on the post-test (Mean=17.5 out of 30, SD=9.9) than the virtual manipulative (Mean=16.0, SD=9.7) and the control groups (Mean=15.5, SD=8.5).

To test the null hypothesis for the difference of post-test scores across instructional conditions, we examined 95% confidence intervals for the means. We found overlaps of confidence intervals (Mean±1.96 SE) across instructional conditions of both grade bands (see error bars in Figures 9–4 and 9–5). Therefore, we further investigated the test statistic for the difference between two means. We found a significant difference at the 5% level between post-test average scores only in third- and fourth-grade physical manipulative and control groups, that is, the 95% confidence interval for the difference between the means of these two groups (0.6, 7.6) did not contain zero.

The findings from students' post-test average total scores indicate that students in all instructional conditions of both grade bands learned to represent and translate equations into different representations, solve one-variable linear equations, and check the solutions. Nevertheless, the third- and fourth-grade physical manipulative group significantly outperformed the control group on the post-test. When comparing post-test performance by grade band, fifth- and sixth-grade performance was lower than third- and fourth-grade performance. A possible explanation for this might be that the fifth- and sixth-grade content was more challenging than the third- and fourth-grade content. In fact, according to the Finnish National Core Curriculum for Basic Education 2014, the content taught in the fifth- and sixth-grade studies is taught in seventh to ninth grades (FNBE 2015, 236, 375). Based on fifth and sixth graders' post-test response, there is evidence of their equation concepts understanding. A fair number of them showed that they used mathematical operations taught during the intervention for solving equations and were able to arrive at the correct solutions. However, they did not receive full scores because of their incomplete steps of solving equations or arithmetic mistakes.

Figure 9-4. Third- and fourth-grade post-test average total scores (out of 24) by instructional condition (error bars = ± 1.96 SE)

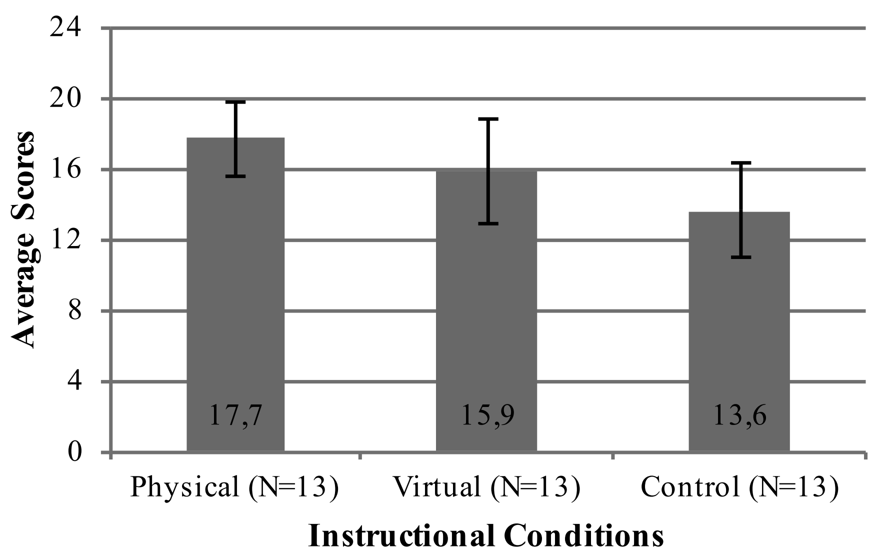
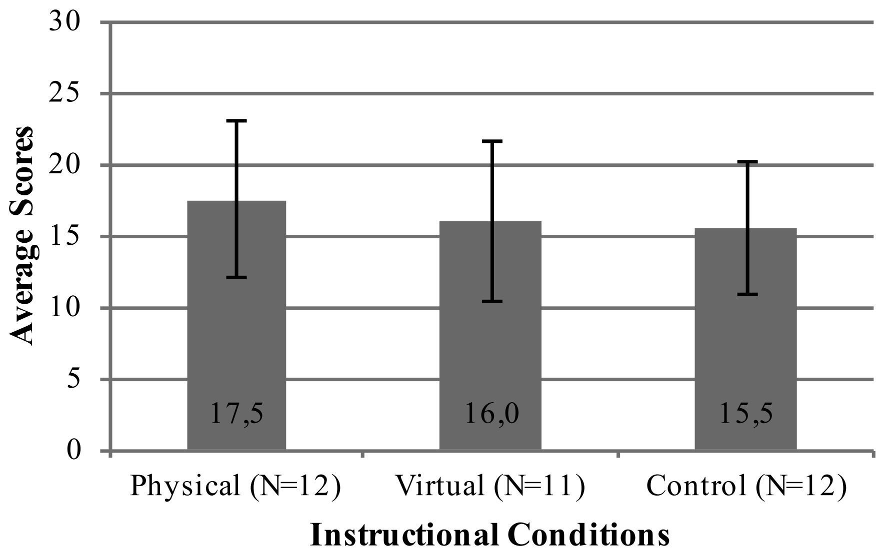


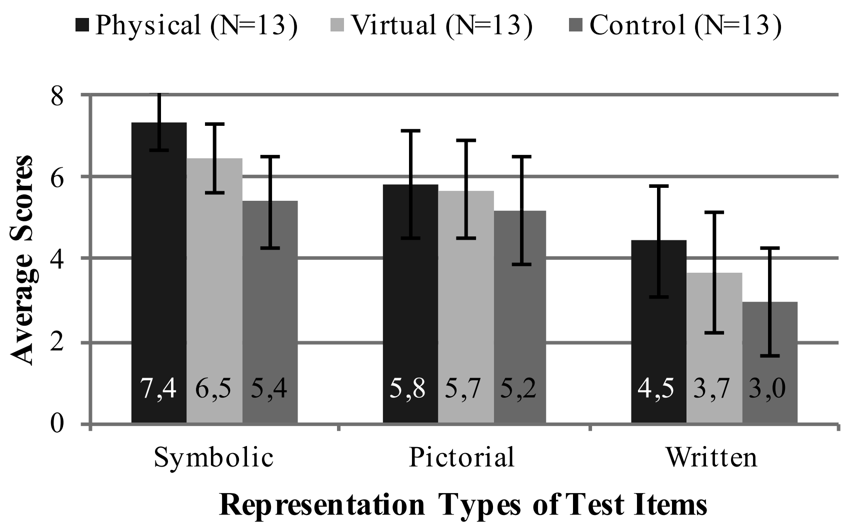
Figure 9-5. Fifth- and sixth-grade post-test average total scores (out of 30) by instructional condition (error bars = ± 1.96 SE)



In addition, we also investigated students' post-tests in three separate sections of different representation types (mathematical symbolic, pictorial, and written) to identify the influence of each instructional condition on students' performance within each post-test section. Figure 9-6 shows that the third- and fourth-grade physical manipulative groups performed best in all eight-full-score sections (Symbolic: Mean=7.4, SD=1.3; Pictorial: Mean=5.8, SD=2.4; Written: Mean=4.5, SD=2.5), relative to the virtual manipulative (Symbolic: Mean=6.5, SD=1.6; Pictorial: Mean=5.7, SD=2.2; Written: Mean=3.7, SD=2.8) and the control groups (Symbolic: Mean=5.4, SD=2.0; Pictorial: Mean=5.2, SD=2.4; Written: Mean=3.0, SD=2.4). As shown in Figure 9-7, even though the fifth- and sixth-grade physical manipulative groups did not perform best in every ten-full-score section, they performed consistently in all test sections (Symbolic: Mean=5.7, SD=2.5; Pictorial: Mean=5.9, SD=3.8; Written: Mean=5.9, SD=4.3), and better than the virtual manipulative (Symbolic: Mean=5.9, SD=2.1; Pictorial: Mean=5.5, SD=3.9; Written: Mean=4.6, SD=4.4) and the control groups (Symbolic: Mean=4.1, SD=3.5; Pictorial: Mean=5.9, SD=2.9; Written: Mean=5.5, SD=4.0).

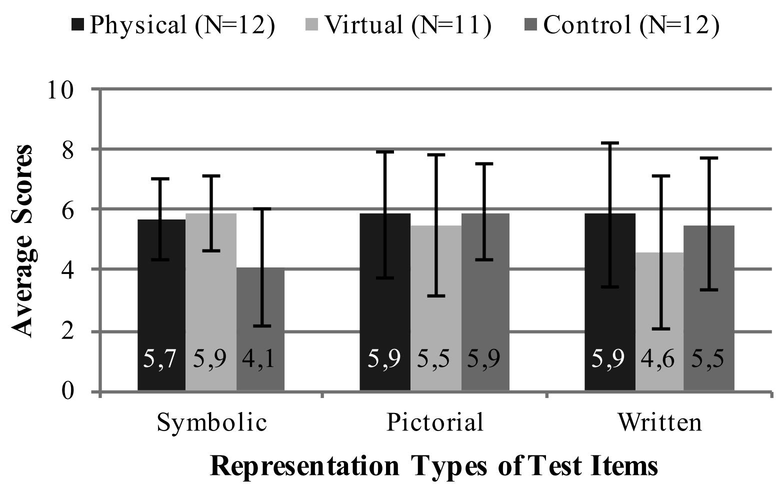
Our findings from the students' performance in different sections of the post-tests are mostly in agreement with the post-test average total scores. The third- and fourth-grade physical and virtual manipulative groups performed better than the control groups in all test sections. However, the difference in the fifth- and sixth-grade post-test performance across test sections was mixed. Although the fifth- and sixth-grade physical manipulative groups' performance in the symbolic section was lower than the virtual manipulative groups', overall, their performance was consistently close to 60% correct in all test sections. In contrast, the virtual manipulative and control groups' performance was inconsistent across test sections.

Figure 9-6. Third- and fourth-grade post-test average scores (out of 8) by representation type of test items across instructional conditions (error bars = ± 1.96 SE)



To discover whether instructional conditions influenced equations-solving strategies on the post-test, we investigated the students' post-test written solutions in terms of their strategies used for solving equations correctly. Their solutions were coded as: 1) trial-and-error substitution of values; 2) reasoning for the unknown; 3) mathematical operations (arithmetic and algebraic); and 4) other strategies. The "other strategies" code was used when students arrived at the correct answer without providing any explanation or steps for solving the equation, or when we were not able to identify their use of strategies. Our analysis did not include the situation where students did not solve the equation or solved the equation but did not arrive at the correct answer. Figure 9-8 shows that third and fourth graders solved 195 equations ($N_{physical}$=68, $N_{virtual}$=65, $N_{control}$=62) correctly by using mostly reasoning for the unknown (50.0% of physical, 55.4% of virtual, and 51.6% of control), followed by mathematical operations and trial-and-error substitution of values, respectively. As shown in Figure 9-9, fifth and sixth graders solved 139 equations ($N_{physical}$=50, $N_{virtual}$=44, $N_{control}$=45)

Figure 9-7. Fifth- and sixth-grade post-test average scores (out of 10) by representation type of test items across instructional conditions (error bars = ± 1.96 SE)



correctly by using strategies from three categories: reasoning for the unknown, mathematical operations, and other strategies. They were more likely to use mathematical operations (80.0% of physical, 79.6% of virtual, and 66.7% of control) than reasoning for the unknown or other strategies.

Our analysis of students' use of strategies to solve equations correctly reveals two main findings. First, in each grade band, the use of strategies for solving equations correctly did not differ overall between instructional conditions. Second, students in all conditions of both grade bands solved equations correctly by using mostly the strategies emphasised during the interventions (reasoning for the unknown in the third- and fourth-grade studies and mathematical operations in the fifth- and sixth-grade studies). Although we found no differences in the strategies used for solving equations correctly across the instructional conditions of each grade band, there was another potentially meaningful difference: On the third- and fourth-grade post-test, mathematical operations (which were never formally taught to third and fourth graders) were slightly more likely to be used for solving equations correctly by the physical manipulative groups than by the two other groups (Figure 9-8). Moreover, three-fifths (8/13) of the physical manipulative groups used this strategy to solve equations correctly at least once, whereas only half (7/13) of the control and one-thirds (4/13) of the virtual manipulative groups did. Likewise, on the fifth- and sixth-grade post-test, the physical and virtual manipulative groups were more likely to use mathematical operations taught during the intervention for solving equations correctly than the control groups (Figure 9-9).

Figure 9-8. Third- and fourth-grade percentage use of different strategies to solve equations correctly (out of 195) by instructional condition

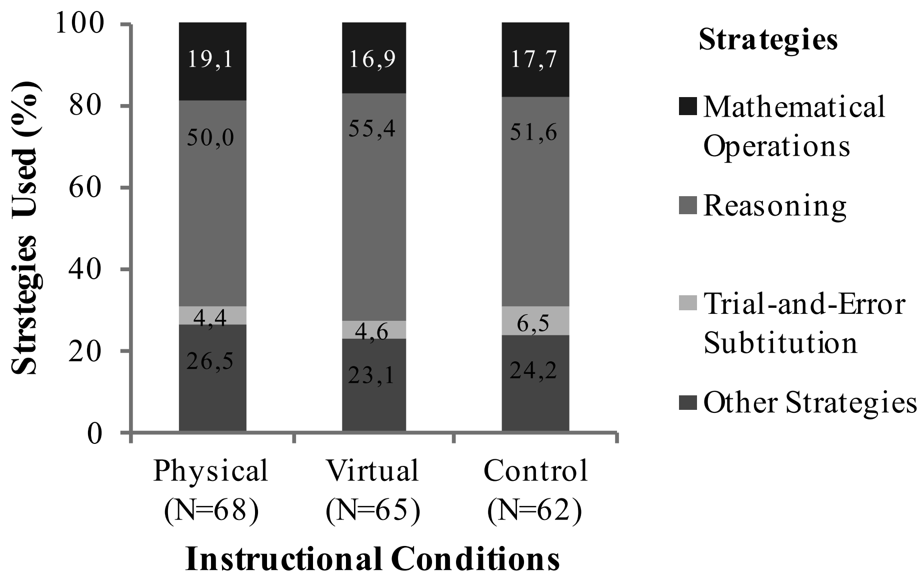
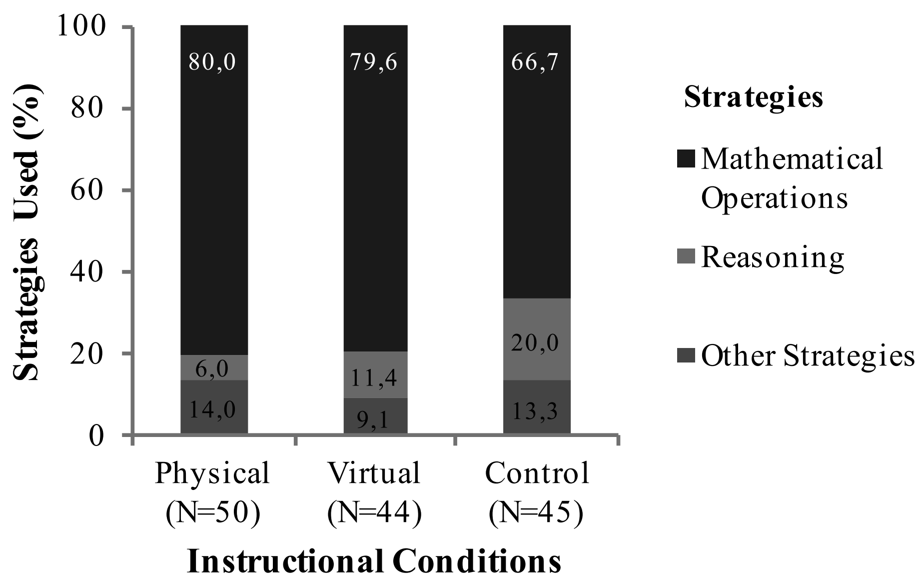


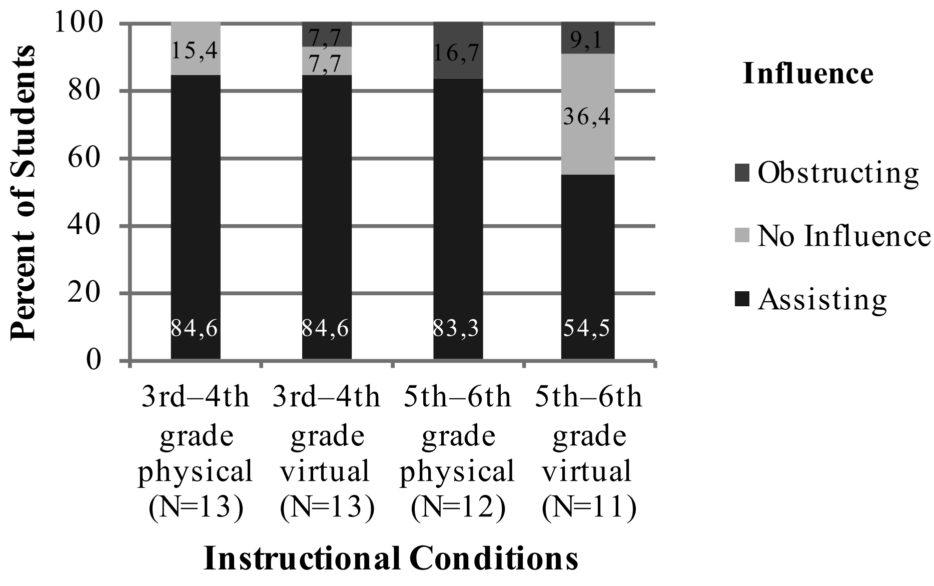
Figure 9-9. Fifth- and sixth-grade percentage use of different strategies to solve equations correctly (out of 139) by instructional condition



Students' self-evaluations

To crosscheck, against their post-test performance, how students assessed their equations learning development, we also concurrently examined their self-evaluation. In terms of learning development, students' self-evaluations generally support their post-test results. Three-fourths (10/13) of third- and fourth-graders in the physical manipulative groups considered themselves better at solving equations after the intervention, whereas less than half of the students in the other two groups (4/13 of virtual and 6/13 of control) considered that they had improved in this regard. Interestingly, although the virtual manipulative groups outperformed the control groups on the post-test, only one-third of them believed that their learning had improved, whereas the majority of them (9/13) considered that their learning had not improved. On the other hand, almost an equal portion of fifth and sixth graders across all three instructional conditions considered that their equations-solving performance had developed (6/12 of control, 7/12 of physical and 6/11 of virtual).

Figure 9-10. Percentage of students' opinions on the influence of manipulatives on their learning by grade band across instructional conditions



Next, we analysed the students' evaluation of the influence of manipulatives on their learning across instructional conditions and grade bands. Figure 9-10 shows that the majority of students in both treatment groups of both grade bands (third and fourth grades: 11/13 of physical and 11/13 of virtual; fifth and sixth grades: 10/12 of physical and 6/11 of virtual) thought that the manipulatives assisted them in learning equations. The results of the students' evaluation correspond to our findings from the post-tests and suggest that manipulatives assisted students in learning equations.

Teachers' pre- and post-interviews

We analysed teachers' pre- and post-interviews to ascertain their viewpoint on how different instructional conditions affected students' learning of equations. After the class interventions, all teachers regarded their physical manipulative group lesson as the most successful. They reasoned that the physical manipulative provided students with a concrete and tactile learning experience and also facilitated students' individual and group languaging. In their opinion, when physically handling the manipulative, students concretely constructed conceptual understanding of: 1) equation equivalence through the balance scale; 2) constants and variables through their distinct representations (number cubes and pawns); and 3) performing the same operation on both sides of the equation through actual action of removing the same elements from both sides of the balance scale. Thus, the physical manipulative groups had a better understanding of equation concepts compared to the other groups. They also believed that students in these groups would perform best on the post-test. Actually, during the pre-interviews, the fifth- and sixth-grade teachers regarded manipulatives as beneficial learning tools for younger students (who are likely to construct their understanding of new concepts through concrete experiences). Nevertheless, after the interventions, the sixth-grade teacher admitted that manipulatives could actually also assist older students (who are likely to have the capability for abstract thinking) in understanding more difficult concepts, such as equations. Moreover, the third- and sixth-grade teachers mentioned that learning how to use the physical manipulative did not take as much of their instructional time as they had expected. Rather, the physical manipulative was straight-forward and generally enabled students to learn and complete the exercise more rapidly than students in the two other groups. This finding agrees with Martin, Svihla, and Smith's findings (2012, 1), but differs from Magruder's (2012, 96).

While the physical manipulative was unanimously regarded as the most successful lesson, the teachers had mixed opinions as to which lesson should be ranked second. The third- and fourth-grade teachers considered their virtual manipulative lesson as the second best and their control lesson as the third best, whereas the fifth- and sixth-grade teachers were not confident about the second and third ranks. Both teachers mentioned that although even low-attaining students in the virtual manipulative groups were able to arrive at the correct solutions to equations during the lessons, they tended to only scroll and try different values for the unknown until arriving at a correct solution rather than developing their understanding. Therefore, these students might not actually understand the equation concepts and would thus perform worse than the control groups on the post-test. Apparently, the teachers' opinions regarding the students' learning achievement correspond to our findings from students' post-tests and self-evaluations. The physical manipulative seems to have had a positive influence on students in both grade bands, whereas the virtual manipulative had noticeable positive benefits only for third- and fourth-graders and appeared to function as an impediment to the development of fifth- and sixth-graders' equation concepts understanding.

Class intervention observations

We analysed the class intervention observations to find empirical evidence for convergent analysis. According to the observations, students in all instructional groups of both grade bands were able to represent the equivalence of the equations in various forms, solve equations, and check their solutions by themselves or with the assistance of their classmates or teachers. Nevertheless, the manipulative groups tended to work more independently, with minimal assistance from the teachers compared with the control groups.

Additionally, we found differences between the physical and virtual manipulatives. The physical manipulative groups had no difficulty in learning how to use the manipulative to model, solve, and check equations. They were more likely to work independently as well as co-operatively. When manipulating the physical manipulative, students usually said aloud (talking to themselves and their classmates) what they were doing or thinking. Consequently, they seemed to develop their understanding of equations gradually, through tactile, visual, and verbal languaging. Simultaneously, their classmates could also see and hear their mathematical thinking. Furthermore, the manipulative allowed the students to solve and check equations without strict procedure.

In contrast, the virtual manipulative was less likely to encourage verbal languaging and co-operative learning. Similar to previous research results (Moyer-Packenham et al. 2013, 36), the virtual manipulative groups tended to work silently and individually, especially when each student had their own iPad. They were more likely to hold the iPad for themselves instead of sharing. Consistent with Magruder's findings (2012, 101) and our teachers' interviews, a number of students seemed to manipulate the virtual manipulative by merely scrolling and trying until arriving at the correct solutions. Moreover, the applet's operational procedure appeared to be relatively complicated and inflexible.

Although students were able to model equations using the applet, several of them had difficulty in learning how to use it to solve and check equations. Consequently, some of them became confused and frustrated. Our findings from the class observations shows the benefits of the physical manipulative and demonstrates that the virtual manipulative functioned as a hindrance to the students' conceptual understanding, verbal languaging, and co-operative learning.

Discussion

Taken together, our convergent analyses demonstrate that students in languaging-based classrooms, across three instructional conditions, in both grade bands, learned to: 1) represent and translate equations into various forms (verbal and written, pictorial, and mathematical symbolic); 2) solve one-variable linear equations; and 3) check the solutions. These findings suggest that students in each instructional condition had developed their representational fluency, which indicated their understanding of equation concepts (NRC 2001, 119). As stated in the earlier literature review, the key to learning of mathematical concepts resides in assisting students in linking concrete and abstract symbolic representations of the same mathematical concepts. In this study, it was the languaging-based instruction that assisted students in classes, with or without manipulatives, in learning equation concepts.

In addition to languaging-based instruction, both manipulatives appeared to facilitate students' development of representational fluency and equation concepts understanding. Overall, both manipulative groups performed better than the control group on the post-tests, where no one had any access to manipulatives. This finding contradicts the claim, mentioned in previous studies, that students tend to over-rely on manipulatives without making connections to the mathematical concepts represented (Magruder 2012, 101; Uttal et al. 2013, 6). Furthermore, we found evidence that the physical manipulative-based instruction is superior to the two other instructional conditions for improving students understading of equation concepts. Students in the physical manipulative groups outperformed their classmates on the post-tests. Likewise, our findings from the students' self-evaluations, the teachers' interviews, and the classroom observations also reveal the positive impact of the physical manipulative on students' conceptual understanding, languaging, and co-operative learning. These findings—on the superiority of the physical manipulative over the virtual manipulative—do not support the previous studies that reported that virtual manipulatives are as beneficial as physical manipulatives to mathematics learning (Moyer-Packenham et al. 2013, 37; Suh and Moyer 2007, 156). This contradictory result may be because, in our study, a number of students manipulated the virtual manipulative in a rote procedural manner to get the correct solutions. Moreover, students were less likely to verbalise their mathematical thinking. These two factors may have negatively affected students' understanding of equation concepts.

In summary, the evidence from this study suggests that when making a connection between various representations constructed through manipulatives and mathematical symbols of the same concept, manipulative-based instruction is more likely to promote students' mathematical concepts understanding. This is consistent with previous research results (Suh and Moyer 2007; Teck 2013). Additionally, our findings support those of other studies in which manipulatives appear to assist students of any age (at any cognitive development level) in developing their understanding of new concepts (McNeil and Uttal 2009, 138).

The presented research results need to be interpreted with caution however, due to some of the inherent limitations—the most obvious of which being the nature of this research as an empirical study conducted in the real contexts of the classroom rather than a laboratory environment. However, the results of research conducted in an authentic teaching and learning context may have provided a better understanding of the real world compared with the findings of research carried out in a laboratory environment. Second, in spite of the teachers' similar qualifications, their different backgrounds and experience as well as their freedom to adjust their lessons may have affected the research results. However, we believe that this had no critical influence on our findings because in each classroom the same teacher taught the same content under all of the instructional conditions. Third, teachers and students may have acted unusually when being observed and video-recorded. Nonetheless, being in a familiar environment (one's own classroom) would likely help them to act more naturally. Fourth, the explanation on the post-test instructions and the encouragement provided during the tests may have had some influence on the students' post-test performance. Still, the explanation and encouragement were, in fact, necessary for students to gain a toehold because the test items were distinctly different from normal school tests and some students became nervous about taking a test after one 45–minute lesson. Fifth, because there was no pre-test before the class interventions, one could argue that the post-test results may have been skewed by the differences in the students' prior mathematics performance levels. However, each class teacher randomly assigned an equal number of students with different prior mathematics performance to each instructional group and so the concern regarding skewed post-test results could be ruled out. Lastly, when conducting cross-sectional case studies, a trade-off between breadth and depth of the study is an unavoidable issue. Due to our limited resources and time as well as the school's constraints (e.g. the number of students per class per teacher), the sample size was rather small and the duration of each class intervention was relatively short. As a result, it is difficult to extend these research findings to other educational contexts.

Conclusion and implications

Our research results highlight the benefits of manipulatives in classrooms for mathematical concepts understanding. These research findings not only provide implications for practice but also for policy-making and future research.

Regarding the question of whether manipulatives should be adopted into practice: our findings support the recommendations, mentioned in our literature review, regarding how to benefit from manipulatives; We recommend that manipulatives be used for facilitating students' understanding of new mathematical concepts. Additionally, manipulatives should be used to assist students in developing their representational fluency (i.e. making a connection between concrete representations constructed through the manipulatives and mathematical symbols of the to-be-learnt concepts) through languaging.

Two implications for policy can be drawn from the presented research results. Our first recommendation resonates with the mathematics instruction objectives of the Finnish National Core Curriculum for Basic Education (FNBE 2015, 128, 235): mathematics curricula should encourage instruction utilising manipulatives in collaboration with languaging to enhance students' representational fluency, which leads to their understanding of mathematical concepts. Our second recommendation is that teacher training should prepare pre- and in-service teachers to effectively benefit from manipulatives.

Despite the fact that this research provides valuable insights into the benefits of manipulatives in classrooms for equation concepts understanding, the limitations of this research make it difficult to generalise our findings to other classroom settings. Therefore, future studies should: investigate larger sample sizes, employ a longer period of class intervention, and add pre- and delay-tests to the research design. During the post-interviews, three out of four teachers mentioned that they plan to use both physical and virtual manipulatives to teach equations in the future. Therefore, it would be valuable to add another treatment group using physical and virtual manipulatives to further research on this topic. Furthermore, to better understand how the mathematics classroom can fully benefit from manipulatives, future research should consider investigating the benefits of manipulatives for diverse learners, different educational levels, and other mathematics content.

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References

Attorps, I. "Mathematics Teachers' Conceptions about Equations". PhD diss., University of Helsinki, 2006.

Australian Curriculum, Assessment and Reporting Authority. *The Australian Curriculum: Mathematics F-10. 2015. www.australiancurriculum.edu.au/download/f10*

Bauersfeld, H. "Language Games in the Mathematics Classroom: Their Function and Their Effects". The Emergence of Mathematical Meaning: Interaction in Classroom Cultures. Eds. P. Cobb and H. Bauersfeld. Hillsdale, New Jersey: Lawrence Erlbaum Associates, Inc., 1995: 271–94.

Common Core State Standards Initiative. Common Core State Standards for Mathematics. 2010. www.corestandards.org/wp-content/uploads/Math_Standards1.pdf

FNBE (Finnish National Board of Education). *Perusopetuksen Opetussuunnitelman Perusteet 2014*. Tampere: Juvenes Print-Suomen Yliopistopaino, 2015.

FNBE (Finnish National Board of Education). *International Comparisons of Some Features of Finnish Education and Training.* 2012. www.oph.fi/download/141160_international_comparisons_of_some_features_of_finnish_education_and_training_2011.pdf

Goldin, G., and N. Shteingold. "Systems of Representations and the Development of Mathematical Concepts". *The Roles of Representation in School Mathematics NCTM Yearbook 2001.* Eds. A. Cuoco and F. Curcio. Reston, Virginia: NCTM, 2001: 1–23.

Joutsenlahti, J., and P. Kulju. "Kieliteoreettinen Lähestymistapa Koulu- matematiikan Sanallisiin Tehtäviin ja Niiden Kielennettyihin Ratkaisuihin". *Toisensa Kohtaavat Ainedidaktiikat. Ainedidaktiikan Symposiumi, Tampere, 2009.* Eds. E. Ropo, H. Silfverberg, and T. Soini. Tampere: Tampereen yliopisto, 2010: 77–89.

Joutsenlahti, J., and K. Rättyä. "Kielentämisen Käsite Ainedidaktisissa Tutkimuksissa". *Rajaton Tulevaisuus - Kohti Kokonaisvaltaista Oppimista.* Eds. M. Kauppinen, M. Rautiainen, and M. Tarnanen. Jyväskylä: Jyväskylän yliopistopaino, 2015: 45–61.

Lesh, R., M. Landau, and E. Hamilton. "Conceptual Models in Applied Mathematical Problem Solving Research". *Acquisition of Mathematics Concepts and Processes. Eds. R. Lesh and M. Landau. New York: Academic Press, 1983: 263–343.*

Magruder, R. "Solving Linear Equations: A Comparison of Concrete and Virtual Manipulatives in Middle School Mathematics". PhD diss., University of Kentucky, 2012.

Marshall, L. and P. Swan. "Exploring the use of mathematics manipulative materials: Is it what we think it is?" *The EDU-COM 2008 International Conference. Sustainability in Higher Education: Directions for Change.* Perth, 2008: 338–50.

Martin, T., V. Svihla, and C. Smith. "The Role of Physical Action in Fraction Learning". *Journal of Education and Human Development 5, 1 (2012): 1–17.*

McNeil, N., and L. Jarvin. "When Theories Don't Add Up: Disentangling the Manipulatives Debate". Theory into Practice 46, 4 (2007): 309–16.

McNeil, N., and D. Uttal. "Rethinking the Use of Concrete Materials in Learning: Perspectives from Development and Education"**.** *Child Development Perspectives 3, 3 (2009): 137–39.*

Moyer-Packenham, P., J. Baker, A. Westenskow, K. Anderson, J. Shumway, K. Rodzon, and K. Jordan. "A Study Comparing Virtual Manipulatives with Other Instructional Treatments in Third- and Fourth-Grade Classrooms". Journal of Education 193, 2 (2013): 25–39.

NRC (National Research Council). Adding It Up: Helping Children Learn Mathematics. Eds. J. Kilpatrick, J. Swafford, and B. Findell. Washington DC: National Academy Press, 2001.

OECD (Organisation for Economic Cooperation and Development). Education Policy Outlook Finland. 2013. www.oecd.org/edu/EDUCATION%20POLICY%20OUTLOOK%20FINLAND_EN. pdf

Ojose, B., and L. Sexton. "The Effect of Manipulative Materials on Mathematics Achievement of First Grade Students". *The Mathematics Educator 12, 1 (2009): 3–14.*

Puchner, L., A. Taylor, B. O'Donnell, and K. Fick. "Teacher Learning and Mathematics Manipulatives: A Collective Case Study about Teacher Use of Manipulatives in Elementary and Middle School Mathematics Lessons". School Science and Mathematics 108 (2010): 313–25.

Suh, J., and P. Moyer. "Developing Students' Representational Fluency Using Virtual and Physical Algebra Balances". Journal of Computers in Mathematics and Science Teaching 26, 2 (2007): 155–73.

Swain, M. "Languaging, Agency and Collaboration in Advanced Second Language Proficiency". Advanced Language Learning: The Contribution of Halliday and Vygotsky. Ed. H. Byrnes. London: Continuum, 2006: 95–108.

Teck, W. "Promoting Conceptual Understanding of Algebra Using Virtual Manipulative: A Study of Form 4 Students". Jurnal Penyelidikan IPG KBL 11 (2013): 1–14.

Uttal, D., M. Amaya, M. Maita, L. Hand, C. Cohen, K. O'Doherty, and J. DeLoache. "It Works Both Ways: Transfer Difficulties between Manipulatives and Written Subtraction Solutions". Child Development Research 2013: 1–13. http://dx.doi.org/10.1155/2013/216367

Part III: Poetic Language