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A Balancing Act

An operational understanding of the equal sign can hinder learning its relational meaning.

By Henry Borenson

Glancing at Chandler's *Semiotics: The Basics* (2007), I discovered that, generally speaking, signs have more than one meaning, depending on the context. Consider, for example, the meaning of the minus sign in the expression 2^{-3}; it certainly does not mean subtraction. Similarly, the equal sign has various meanings, depending on the context in which it is used (Molina, Castro, and Castro 2009). The assumption that the equal sign can have only one meaning has led some educators to state that "it does *not* mean that the answer comes next."

The use of the equal sign to indicate the *unique* numerical result of the sequence of computations that precede it (the "calculator use" of the equal sign) is a valid and necessary use of this sign (Ginsburg 1996; Falkner, Levi, and Carpenter 1999; Seo and Ginsburg 2003).

However, understanding the relational meaning of the equal sign also is essential for success in mathematics, and particularly in algebra. Whereas such equations as $3x + 5 = 26$ can be understood knowing only the operational meaning of the equal sign (because 26 can be considered the result of the operations that appear to the left of the equal sign), examples such as $4 + 3 = \_\_ + 2$ and $4x + 2 = 2x + 6$ cannot (Kieran 1992). In these examples, the equal sign indicates equivalence between two sets of expressions, each one of which includes one or more operations within it.

Is the relational meaning of the equal sign self-evident to students? Do elementary school students intuitively understand the relational meaning of the equal sign? In a survey of 752 students in grades 1–6, Falkner, Levi, and Carpenter (1999) found that only 5 percent of the students provided the correct response to the question $8 + 4 = \_\_ + 5$. As the authors noted, the equal sign usually is presented to elementary school students only "at the end of an equation, and only one number comes after it" (p. 233).

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We can therefore conclude that the relational meaning of the equal sign is not something that students find intuitive or self-evident, nor is it an understanding that naturally follows from knowing the operational meaning of the equal sign.

This result should not be too surprising if we consider that the meanings of signs are arbitrary, that they are determined by convention (Chandler 2007; Carpenter, Franke, and Levi 2003). Convention *might* have dictated, for example, that the equal sign would be the symbol used specifically to indicate the answer to the set of calculations preceding it, and that it should be read only from left to right (see Seo and Ginsburg 2003, for an alternative symbol to the equal sign for displaying the answer to a set of calculations). For equivalence, mathematical convention *might* have provided a different symbol, such as ≡ or ⇔. Neither of these possibilities was realized. Instead, the equal sign is used to indicate both meanings.

However, a computer cannot function if the same symbol has more than one meaning. Hence, in the computer language C++, the symbol = = (two equal signs next to each other) is the symbol for equivalence. A single equal sign is used for assignment of a value—the operational meaning of the equal sign—except

A six-year-old research-lab student used numbered cubes to represent 5 + 2 = __ + 3 on a diagram of a balance scale.

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that the order is reversed. In **figure 1**, the computer must first evaluate the given expression before it can consider if it is equivalent to 24.

These examples illustrate the folly of expecting students who know the operational meaning of the equal sign to intuit the sign's relational meaning. It is simply a matter of convention that in human mathematics, the equal sign has both operational and relational, or equivalence, meanings.

Triggering the operational meaning

Let's consider the following scenario with a class of elementary school students who have not yet been introduced to the relational meaning of the equal sign. The teacher presents a problem such as 8 + 4 = __ + 5 and asks students what they think the missing number is and to justify their answers. On the basis of students' prior experience with the equal sign, a large majority will respond with the answer 12 because "the answer follows the equal sign." That is, the equal sign will tend to trigger the operational definition. If asked, "What about the plus five?" they have a ready answer: "Maybe they just put it there to confuse us" or "to distinguish what is important," as is often done with extra information provided in story problems (Carpenter, Franke, and Levi 2003, p. 10).

Even if a few students in the class say that the answer is seven ("The equal sign means that both sides have the same value"), the rest of the class usually will not readily accept that point of view because they will not be aware that this is a valid use of the equal sign as established by convention. Once students convince themselves that the correct response to 8 + 4 = __ + 5 is 12, and they see the majority of the class confirming their reasoning, it will be difficult to dissuade them. Indeed, many students "may cling tenaciously to the conceptions they have formed about how the equal sign should be used" (Carpenter, Franke, and Levi 2003, p. 12).

This instructional problem will be compounded if the teacher, in trying to teach the relational meaning of the equal sign, says that "the equal sign does *not* mean that the answer comes next." What then are the students to think? They know how they have used the equal sign countless times. Is the teacher asking them to discard their prior understanding? Resistance sets in.

Effectively introducing the relational meaning of the equal sign requires a strategy, first, to avoid triggering the operational meaning of the equal sign and, second, to validate—or at least not invalidate—students' prior experience with the operational sense of the equal sign.

Effectively introducing the relational equal sign

One effective approach for introducing the relational meaning of the equal sign is initially to teach students the concept of balance and only later relate that learning to the equal sign. This is the approach taken by Mann (2004), who cleverly has third-grade students use their bodies to

FIGURE 1

A snippet from the program C++ shows the operational use of the equal sign in line 1 and its relational use in line 2.

```
1.  x = (5 + 1) * (5 - 1) ;
2.  if ( x == 24 )
3.      printf ('Yes, 24');
4.  else printf ('Not 24');
```

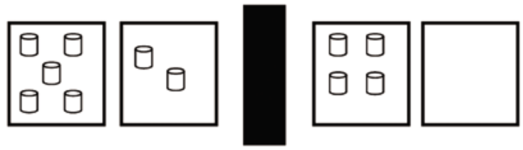
simulate a seesaw so that they can learn the concept of balance. She places imaginary weights on one or both sides of each student, and the students move accordingly, thereby reflecting whether they are in balance. The students demonstrate that they have learned how to maintain balance by adding or taking away the same weight from each side. Mann then places an equal sign on the board and asks students how this sign relates to what they have been doing. Students pick up the hint. For example, Sari responds, “If you have 3 oranges and 2 apples on one side of the equals sign, just like we had on the seesaw, and 3 oranges and 2 apples on the other side of the equals sign, you have the same on both sides” (p. 68).

On their own, students are able to transfer the concept of balance to the equal sign. At this point, they are ready to consider such problems as $4 + 5 = \_\_ + 3$. If any student is stuck, he or she can be asked to visualize the given numbers as weights on the imaginary seesaw. Students also can be asked to make a drawing of the problem or to represent the problem concretely on a drawing of a balance scale (see **fig. 2**).

FIGURE 2

The Sherman and Bisanz study (2009, p. 91) used nonsymbolic and symbolic representation of equivalence problems.

Nonsymbolic Representation



$5 + 2 = 4 + \_\_\_\_\_\_$

Symbolic Representation

A computer cannot function if the same symbol has more than one meaning.

Research conducted by Sherman and Bisanz (2009) confirms the value of first nonsymbolically introducing the concept of equality. The second graders in their study were able to apply what they had learned (about nonsymbolic equivalence in a twenty-minute lesson) to the equal sign symbolic notation one week later, even when no attempt was made to relate their prior learning to the equal sign. **Figure 2** shows how Sherman and Bisanz represented nonsymbolic and symbolic equivalence. The results led these researchers to draw the following conclusion:

> Having children learn to reason about equivalence problems in a nonsymbolic format would be a relatively easy, inexpensive intervention that could have a positive impact on children's ability to solve similar problems presented symbolically. (p. 97)

Maintaining the balance

Elementary school students must understand that the equal sign can be used to indicate equivalence between two expressions. If teachers do not carefully think through their instruction, the students' prior valid operational understanding of the equal sign will pose an obstacle to learning the relational meaning.

By first teaching the concept of equivalence nonsymbolically, using the balance model or using concrete objects, and only afterward relating that learning to the symbolic notation, we can provide young students with a successful introduction to the relational meaning of the equal sign.

Common Core Connection

1.OA.D.7

REFERENCES

Carpenter, Thomas, Megan L. Franke, and Linda Levi. 2003. *Thinking Mathematically: Integrating Arithmetic & Algebra in the Elementary School*. Portsmouth, NH: Heinemann.

Chandler, Daniel. 2007. *Semiotics: The Basics.* 2nd ed. London: Routledge.

Falkner, Karen, Linda Levi, and Thomas P. Carpenter. 1999. "Early Childhood Corner: Children's Understanding of Equality: A Foundation for Algebra." *Teaching Children Mathematics* 6 (December): 232–36.

Ginsburg, Herbert P. 1996. "Toby's Math." In *The Nature of Mathematical Thinking,* edited by Robert J. Sternberg and Talia Ben-Zeev, pp. 175–202. Mahwah, NJ: Lawrence Erlbaum Associates.

Kieran, Carolyn. 1992. "Learning and Teaching of School Algebra." In *Handbook of Research on Mathematical Teaching and Learning*, edited by Douglas A. Grouws, pp. 390–419. New York: Macmillan.

Mann, Rebecca L. 2004. "Balancing Act: The Truth behind the Equals Sign." *Teaching Children Mathematics* 11 (September): 65–69.

Molina, Marta, Encarnación Castro, and Enrique Castro. 2009. "Elementary Students' Understanding of the Equal Sign in Number Sentences." *Electronic Journal of Research in Educational Psychology* 17 (7): 341–68.

Seo, Kyoung-Hye, and Herbert P. Ginsburg. 2003. "'You've Got to Carefully Read the Math Sentence. . . .' Classroom Context and Children's Interpretation of the Equals Sign." In *The Development of Arithmetic Concepts and Skills*, edited by Arthur J. Baroody and Ann Dowker, pp. 161–86. Mahwah, NJ: Lawrence Erlbaum Associates.

Sherman, Jody, and Jeffrey Bisanz. 2009. "Equivalence in Symbolic and Nonsymbolic Contexts: Benefits of Solving Problems with Manipulatives." *Journal of Educational Psychology* 101 (1): 88–100.

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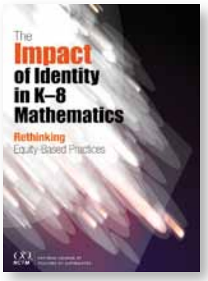

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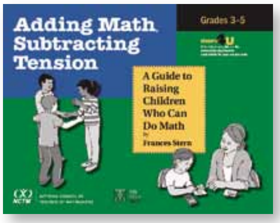

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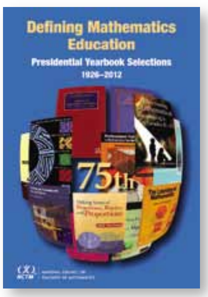

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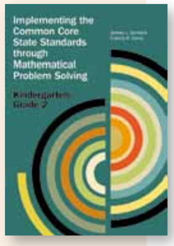

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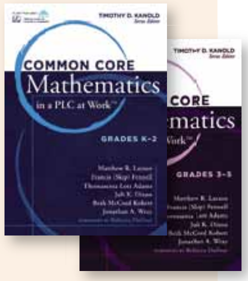

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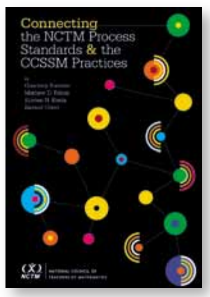

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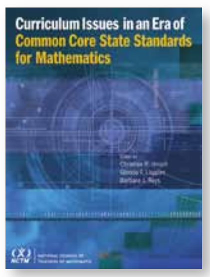

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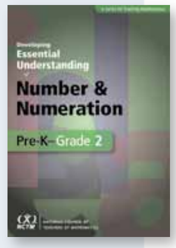

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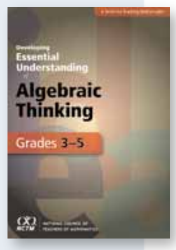

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