A STUDY OF A MANIPULATIVE APPROACH TO TEACHING LINEAR EQUATIONS TO SIXTH GRADE STUDENTS

A PROFESSIONAL PAPER

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF SCIENCE IN SCIENCE EDUCATION

IN THE GRADUATE SCHOOL OF THE

TEXAS WOMAN'S UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

BY

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DENTON, TEXAS

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TEXAS WOMAN'S UNIVERSITY
DENTON, TEXAS

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Date

To the Dean for Graduate Studies and Research:

I am submitting herewith a professional paper written by Jennifer Barclay entitled "A Study of a Manipulative Approach to Teaching Linear Equations to Sixth Grade Students." I have examined the final copy of this professional paper for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science, Science Education, with a major in Mathematics.

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Dr. Frances Thompson, Major Professor

We have read this professional paper and recommend its acceptance:

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Accepted

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Dean for Graduate Studies and Research

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In the past, algebra was introduced to students in the eighth grade or above who demonstrated some degree of prowess in mathematics. Teaching simple linear equations is now one of the sixth grade essential elements prescribed by the Texas Education Agency, and at least 75% of the children exiting elementary school mathematics programs progress to the study of algebra. (Post, 384) And yet so many students at this age are still primarily concrete learners that an abstract approach is simply not pragmatic. Many students getting their first frustrating abstract taste of algebra find it not only difficult to swallow and digest, but very threatening. A large number of students consequently develop a fear and growing dread of the upcoming algebra class. Past learning experiences, retained as memories, alter our perspectives in new learning situations. (Braun, 102) Having fun and experiencing success early in linear equations may provide the students with the confidence they will need to be successful in later algebra courses.

Purpose of Study

In this paper a study based on a manipulative game approach to teaching sixth grade students linear equations will be presented. Several hypotheses were considered in the study: (1) all students ages 11 to 13 are intellectually capable of solving equations such as $2(X + 8) = 5X + 1$, and (2) at the sixth grade level, regular education students and learning different students taught with manipulatives will retain concepts learned at the same rate as students taught with manipulatives in the gifted/talented class.

Definitions for Study

(1) Manipulatives: Manipulatives will be defined in this study as objects which represent mathematical ideas that can be abstracted through physical involvement with the objects. (Young, 12) The student is able to feel, touch, handle, and move the objects. They may be real objects which have social application in everyday affairs, or they may be objects which are used to represent a real object or an idea. (Grossnickle, 162)

(2) HANDS-ON EQUATIONS: The instructional materials used in the study are called HANDS-ON EQUATIONS designed and developed by Dr. Henry Borenson. HANDS-ON EQUATIONS uses pawns, numbered cubes, and a balance beam as the manipulatives governed by rules known as "legal moves". Students manipulate both sides of the equation, while maintaining balance, to simplify the equation until the solution is obtained. For a more detailed description of the system, please refer to Chapter III and Appendix C.

(3) Concrete: Characterized by or belonging to immediate experience of actual things or events. In this study 'concrete' implies dealing with a tangible object relating to a specific task. (Webster, 232)

(4) Pictorial: Pertaining to diagrams or drawings that represent concrete objects or actions.

(5) Abstract: Something disassociated from any specific instance or object but that represents a family of ideas or events. (Webster, 5)

(6) Kinesthetic/Tactile: Relating to the sense of touch. (Webster, 1177)

(7) Multisensory: Involving many senses --visual, auditory, tactile.

(8) Learning Different Student: A child who, despite average or above average intelligence, does not respond to normal instruction with achievement reflective of his intelligence. This includes students with dyslexia, dysgraphia, and dyscalcula. The terms 'learning different' and 'learning disabled' are frequently used interchangeably.

(9) Attention Deficit Hyperactivity Disorder: The core features of the disorder fall into three categories of inattention, impulsivity, and hyperactivity. Associated features may include oppositional behavior, conduct disorders, emotional difficulties, and cognitive and learning disabilities. (TEA, 3)

(10) At-Risk Student: Student identified at risk of becoming discouraged and dropping out of school. At-risk students may qualify for remedial programs based on the following state criteria:

1. Unsatisfactory on readiness or achievement assessment instrument administered at the beginning of the school year.
2. Failed any section of most recent Texas Assessment of Academic Skills (TAAS).
3. Identified Limited English Proficient (LEP).
4. Unexcused absences.
5. Nonhandicapped residing in residential placement facility outside parents' district.
6. Failed to meet promotion requirements in any previous grade.

(Hammit, 1)

(11) Gifted/Talented Students: The Texas Education Agency defines these as students who, by virtue of outstanding mental abilities, are capable of high performance. The student may demonstrate, singly or in combination, above-average achievement or potential in such areas as general intellectual ability, specific subject matter aptitude, ability in creative and productive thinking, and leadership ability. (Red Oak, 3)

(12) Retention: An ability to retain things in mind; preservation of the aftereffects of experience and learning that makes recall or recognition possible. (Webster, 980)

Procedure of Study

The study involved four regular education classes and one gifted/talented class of sixth grade students. All classes received the same instruction and used the manipulatives from HANDS-ON EQUATIONS. Students completed a pretest, six lessons from HANDS-ON EQUATIONS (Level I), a posttest, a three week retention test, and a six week retention test. Results from the four tests were used to assess the level of concept mastery and retention.

Limitations

(1) One limitation of this study was the size of the group involved. There were 123 students who completed all phases of the study, but this is a small sample of sixth grade students to investigate.

(2) The population of the school was also not ethnically reflective of a larger school district nor socio-economically reflective of a larger city. The district

in which the study was conducted is composed of 15% minority students while a large district in an adjacent county is 51% minority students.

(3) Another characteristic of this study that may be seen as a limitation is that all the students received the same instruction. There was no control group. This was not a comparative study in the sense of having a control group and an experimental group.

CHAPTER II

REVIEW OF LITERATURE

Many factors must be considered in education. A few of these factors are: the psychological development of the child, the concept being taught, the child's reaction to different forms of instruction, and the child's ability to remember and apply the concepts he has learned. A review of these factors follows.

Goals of Mathematics

Effective mathematics instruction is primarily concerned with concept formation as opposed to the memorization of facts. The mental processes involved in concept formation are much more complex than those associated with the memorization of a mass of isolated details. (Reys. 551) The teacher must find ways of teaching other than "show and tell" in order for children to remember mathematical ideas. (Copeland, 356) Instruction must go beyond simple knowledge to understanding.

Developmental Psychology

Jean Piaget, one of the foremost researchers in child development. classifies learners into four stages of cognitive development: "sensorimotor (birth to about two years). preoperational (two to about seven years). concrete operational (seven to about 11 years), and formal operations (about 11 years to adult). the last two of which are most relevant to middle school." (Juraschek.58) The concrete operational child can reason inductively - going from his own experience to a general principle. But he has trouble going the other way - from a general principle to some anticipated experience. He has a hard time imagining things he has never experienced and has an equally hard time with abstract concepts when they are not linked to specific objects. In formal operations, the young person becomes able to manipulate ideas as well as objects and can approach problems systematically. Deductive logic appears. (Bee, 225) Concrete operations are bounded by the immediate and "real"; formal operations are not. With concrete operations, reality dominates thought, but with formal operations, possibility dominates thought. (Juraschek, 58) Unlike earlier stages, which virtually all children go through, formal operations are achieved by only about half of the adolescents. (Bee, 248) A typical middle school class then will most likely contain a few students who are concrete-operational and a few who are formal-operational, with the majority of the the students in some transitional phase between the two stages. (Juraschek, 59)

Manipulatives in Mathematics

Marilyn Suydam and Jon Higgins found in their research of students in grades one through eight that the use of manipulative materials has a higher probability of producing greater mathematical achievement than do non-manipulative sequences. They found that when learning is broken into three stages -- from concrete objects to pictures to symbols, the symbolic treatments are probably at a disadvantage when used without concrete objects or pictorial models. The studies which found pictorial treatments to be superior to symbolic treatments confirm that pictures and diagrams can also be important in designing math lessons. However, pictures are rarely superior to concrete experiences. (Suydam[1], 24) Abstraction is only a sort of trickery and deflection of the mind if it does not constitute the crowning stage of a series of previously uninterrupted concrete actions. The true cause of failure in formal education is therefore essentially the fact that one begins with language (accompanied by drawings) instead of beginning with real materials and action. (Piaget, 104) One major problem in schools is the fact that many children are asked to abstract mathematical ideas before they have the opportunity to experience them in concrete form. (Post, 11)

Algebra and Manipulatives

Elementary algebra crosses the chasm between the arithmetic of everyday life and the abstract symbolism of mathematics proper. (Adler, 79) Educators want to do as much as possible to fill in this chasm between algebra and arithmetic. Former methods of teaching algebra were suited strictly for abstract learners, thus leaving out the younger and/or more concrete learner. Jerome Bruner states that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development." Hence, many younger students can learn some of the basic abstract concepts of algebra through games using manipulatives. Manipulatives motivate students; manipulatives stimulate students to think mathematically; and manipulatives informally introduce "big ideas" in mathematics. (Herbet, 4) But all that glitters is not gold, and all manipulatives on the market are not equally valid.

Robert Reys from his studies suggests the following criteria in selecting manipulative materials:

1. The manipulative should provide a true embodiment of the mathematical concept or idea being explored.
2. The materials should clearly represent the mathematical concept.
3. The materials should be motivating.
4. The materials should be multipurpose if possible (appropriate for several grade levels and different levels of concept formation).
5. The materials should provide a basis for abstraction.
6. The materials should provide for individual manipulation. (Reys, 553)

Marilyn Suydam offers one caution: "Not all children need to use manipulatives for the same amount of time. Prolonged use may keep some children using procedures too simple and inefficient for them. Concern for individual needs must govern the use of manipulative materials." (Suydam[2], 27)

Retention

While memorization of facts is not equivalent to concept formation. "memory gives learning some permanence and also influences the way in which we learn." (Braun, 102) Memory is the capacity to retain information. Learning provides the substance, or content, of memory. In turn, memory of past experiences and their consequences provides the basis for learning in future situations of a similar nature. (Braun, 121)

Performance is determined both by what is remembered and by what is forgotten. Performance is also influenced by factors such as motivation and emotional state. Manipulatives tend to be motivating to students. They aid in holding attention and focus on a concept. Storing information in memory and retrieving it at a later time depend a great deal on attention, the ability to focus on certain information from the environment while ignoring other information. Our senses are constantly being bombarded by the environment -- sights, sounds, odors, tactile sensations are all present simultaneously. Students have different learning styles: some are visual learners, others are auditory, while others are kinesthetic/tactile-oriented learners. This can be a disadvantage in teaching or can be used as an advantage by incorporating as many senses in the learning process as possible. A multisensory approach would be more apt to include all learners as opposed to a method which favored only one learning style. The brain monitors incoming sensory information to some extent, directing attention to one type of sensory input while putting a damper on information that is less important. (Braun, 104)

CHAPTER III

DESCRIPTION OF STUDY

Various aspects of the study included the population of the group, the process of instruction, the system of manipulatives used, and the schedule of instruction. Each aspect is discussed in more detail in this chapter.

Population of Study

The study was conducted at an intermediate school containing grades five and six. The community socioeconomic structure around the school is predominantly white, Anglo-Saxon, and protestant. Parts of the region are rural but overall, it is more of a suburban area. Consequently, the majority of the students in the study were from middle income families with a few students from lower income families and a few from upper income backgrounds. This might appear then as a limited population. However, included in the study were seven resource students who qualify for special education assistance in mathematics, twelve learning disabled students who have various learning differences, and five students who are identified as having attention deficit disorder and/or hyperactivity disorders. Eighty-five percent of the students in this study were white, with fifteen percent of the students being from other backgrounds (Hispanic, Black, Middle Eastern). So although the group was predominantly middle income white students, there were enough other factors to make the study of interest and worth.

Procedure

Ninety-two sixth grade students were involved in this study ranging in age from 11 years to 13 years. These students were grouped heterogeneously in four different classes, and the distribution of students' grades in mathematics from class to class was comparable. Another 31 students were also involved in the study. These 31 were in an additional class and classified as gifted and talented. They received the same instruction as the other 92 students. All but two of the total 123 students had had some previous instruction in simple linear equations. In order to test the amount of previous knowledge retained and to assess the appropriate entry level for instruction, a pretest of 10 items was administered. After the appropriate level of instruction was determined, a series of five lessons was presented. Each student had his own desktop set of manipulatives to maneuver, guided by HANDS-ON EQUATIONS written lessons and instruction by the classroom teacher. At the conclusion of the five lessons, a posttest was administered to assess mastery of concepts. Then followed three weeks of non-algebraic type instruction with no review of linear equations. At the conclusion of the three-week suspended algebraic activity period, a retention test was administered followed by another three-week period with no review and a six-week retention test. Results of the four tests were used to determine the level of concept mastery and retention.

Description of Manipulatives

From the HANDS-ON EQUATIONS Learning System (Appendix C), each student had a desktop set of manipulatives consisting of a laminated picture of

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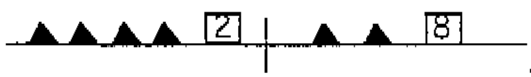
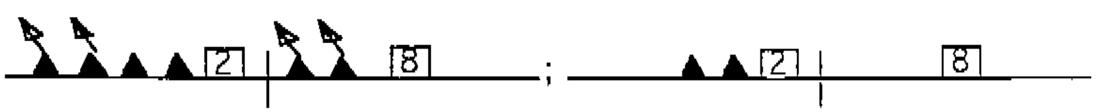
a balance beam, eight blue pawns (similar to chess game pieces) and four red number cubes -- two cubes numbered 0-5 and two cubes numbered 5-10. The instructor had an actual stationary balance beam model on which to set up an oversized set of manipulatives for classroom demonstration.

Each blue pawn represented the variable, designated as "x". The number cubes represented the constants (but not coefficients). The pictorial balance beam represented the idea of a balanced equation. The students started with a balanced system, and the object was to determine the value of the variable while maintaining the balanced system. The representation then for the problem $4x + 2 = 2x + 8$ would be:

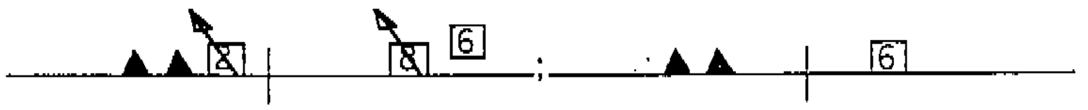


Students learned to simplify the equation by using "legal moves".

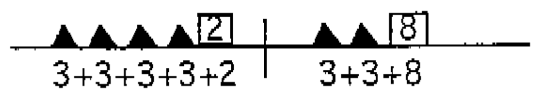
One legal move the students used simply removed the same number of pawns from each side of the balance beam. Since each pawn had the same value, this move would leave the system in balance; thus, $4x + 2 = 2x + 8$ simplified to $2x + 2 = 8$:



The next legal move the students learned was to remove the same amount from the number cubes on each side of the balance beam. Thus, $2x + 2 = 8$ simplified to $2x = 6$:



From this representation, the students determined the value of each pawn to be 3, thus x = 3. The system emphasized the importance of checking the value of x against the original equation set-up to ensure all moves were legal and the correct value of x had been determined. Students set up the original equation and then checked to see if it balanced when each pawn was assigned the value that had been determined.



Since 14 = 14 the system is in balance when x = 3.

<u>Schedule of Instruction</u> (See Appendix B for details of daily lessons.)

I. Pretest

II. Instruction

Day 1 - Students became familiar with the system. Worksheet #1

Day 2 - Students learned to set up equation from actual algebraic notation. Worksheet #2

Day 3 - Students used the legal move for the variable. Worksheet #3

Day 4 - Students used legal moves for variables and number cubes. The teacher introduced subtraction notation. Worksheets #4 & 5.

Day 5 - The teacher introduced notation with parentheses and pictorial representation. Worksheet #6.

III. Posttest

IV. Three-Week Retention Test

V. Six-Week Retention Test

CHAPTER IV

PRESENTATION OF DATA

Data were gathered on each individual student for each test as well as collectively for each class. Demographic information for each student and his/her performance on the pretest, posttest, three-week retention test, and six-week retention test follows. Each test was made up of ten items. The ten items were of the same form from test to test, but were not identical. An individual score of 8 out of 10 (80%) or better was considered mastery of the concept at this level. Classes A, B, C, and D were regular education groups that were judged similar in ability based on individual student mathematics grades. Letters A, B, C, and D were assigned to the classes at the beginning of the year. Students were placed randomly in these classes. Each class of students travels together as a group to their teachers for different subject instruction. Class G/T designates the Gifted/Talented group used in the study for comparison purposes. These were 15 students from a class E and 16 students from a class G that come together one time during the day for mathematics instruction and enrichment.

Students having previous experience (but extent not documentable) with the HANDS-ON EQUATIONS System have been identified with an asterisk (*). All other students had previous instruction with linear equations only at the symbolic level and this instruction occurred four months prior to this study. At that time only equations of the type $x + a = b$, where a and b are integers, was studied . This equation form was represented by items #1 and #2 on each test administered.

Test Results for Classes

The following tables contain descriptions of individual students in each class, as well as their different test scores.

TABLE 1A: TEST RESULTS FOR CLASS A

Student	Sex	Ethn	Days Absent	Special Learner Codes	Pre	Post	3wk	6wk
A-1	M	W	0	SE,AH	30	90	80	100
A-2	F	W	0	R-2	30	100	90	100
A-3	F	W	0	R-2,6	10	100	90	100
A-4	F	W	1	R-2	30	90	100	100
A-5	F	W	0		40	100	100	100
A-6	M	W	0		40	100	100	100
A-7	F	W	0	SE	30	80	70	100
A-8	F	W	1		60	100	100	100
A-9	M	W	0		30	100	100	90
A-10	M	W	1		abs	80	80	90
A-11*	M	W	0	SE,R-2,6 AH	50	100	20	100
A-12	F	W	0	SE	0	100	90	100
A-13	F	W	0		0	100	100	100
A-14	F	W	0	R-2	30	90	90	100
A-15	M	W	0		20	90	90	100
A-16	M	W	0	R-2	40	100	100	90
A-17	M	W	0		30	100	80	100
A-18	M	W	0		20	100	100	100

KEY

Ethnicity

- W - White
- B - Black
- H - Hispanic
- ME - Middle Eastern

* Student had previous instruction with HANDS-ON EQUATIONS

Special Learner Codes

- AH - Attention Deficit Hyperactivity Disorder
- LD - Learning Different
- SE - Special Education
- R - At-Risk
  - R-2 Failed one or more portions of most recent TAAS
  - R-4 Excessive Absences
  - R-6 Retained in a previous grade

TABLE 1B: TEST RESULTS FOR CLASS B

Student	Sex	Ethn	Days Absent	Special Learner Codes	Pre	Post	3wk	6wk
B-1	F	W	0		30	100	100	100
B-2	M	W	1		60	90	100	100
B-3	M	W	0		30	100	100	100
B-4	F	W	1	R-2,6	30	100	100	80
B-5	M	B	0	R-2,4	20	80	70	100
B-6	F	W	0		30	90	80	90
B-7	M	W	0	R-2	20	100	100	100
B-8	M	H	0		30	80	90	100
B-9	F	W	0		20	100	100	100
B-10	M	W	2		0	70	90	90
B-11	F	W	0		20	90	100	100
B-12	F	W	0		40	100	100	100
B-13	M	W	0		30	100	70	80
B-14	F	W	0		30	100	100	100
B-15	F	B	0		20	100	100	100
B-16	F	W	0		40	100	100	100
B-17	M	W	0	R-2	20	100	80	80
B-18	M	W	0		30	100	90	90
B-19	F	W	0		30	100	100	90
B-20	M	W	1		20	100	70	90
B-21	F	W	0	R-2	20	80	90	90
B-22	F	W	0	R-2	20	90	90	90
B-23	F	W	0		30	90	90	100
B-24	M	W	0		50	100	90	90
B-25	F	W	0		0	80	100	100

KEY

Ethnicity
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- ME - Middle Eastern

* Student had previous instruction with HANDS-ON EQUATIONS

Special Learner Codes
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  - R-4 Excessive Absences
  - R-6 Retained in a previous grade

TABLE 1C: TEST RESULTS FOR CLASS C

Student	Sex	Ethn	Days Absent	Special Learner Codes	Pre	Post	3wk	6wk
C-1	M	H	1	LD,R-2	30	100	90	90
C-2	M	W	1	R-2	30	100	80	100
C-3	F	W	0		20	90	90	100
C-4	F	W	0	R-2	20	90	100	90
C-5	M	W	1	LD,R-2	10	100	60	80
C-6	F	W	0		30	100	100	100
C-7	M	H	0	LD,R-2	30	100	100	70
C-8	F	H	0	LD,SE	0	100	100	100
C-9	M	W	0	R-2,AH	30	90	abs	100
C-10	F	W	0	R-2	30	80	100	100
C-11	F	H	0		20	90	100	70
C-12	M	W	0	R-2	30	100	100	90
C-13	F	W	1	R-2	abs	100	90	90
C-14	M	W	0	LD,R-2	20	90	90	100
C-15	F	W	0	R-2	20	80	100	80
C-16	F	W	0	LD,SE	40	100	100	90
C-17	F	W	0	R-2,6	abs	100	100	100
C-18	F	H	1	R-2	20	90	100	90
C-19	M	W	0		30	100	70	90
C-20	M	H	0		30	100	90	100
C-21	F	W	1	LD,R-2	10	100	abs	100
C-22	M	W	0	R-2	30	100	100	90
C-23	F	W	0		20	90	100	90
C-24	F	W	0		20	100	100	100

KEY

Ethnicity
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- H - Hispanic
- ME - Middle Eastern

* Student had previous instruction with HANDS-ON EQUATIONS

Special Learner Codes
- AH - Attention Deficit Hyperactivity Disorder
- LD - Learning Different
- SE - Special Education
- R - At-Risk
  - R-2 Failed one or more portions of most recent TAAS
  - R-4 Excessive Absences
  - R-6 Retained in a previous grade

TABLE 1D: TEST RESULTS FOR CLASS D

Student	Sex	Ethn	Days Absent	Special Learner Codes	Pre	Post	3wk	6wk
D-1	F	W	0	LD	20	100	80	100
D-2	F	W	0	R-2	20	100	80	100
D-3	F	W	0	R-2	10	100	90	100
D-4	F	W	1	LD,R-2	20	100	90	90
D-5	F	W	0		30	100	90	100
D-6	M	H	0	R-2	30	90	70	80
D-7	F	W	2		30	100	90	100
D-8	M	W	0	R-2	30	100	100	100
D-9	M	W	0		30	100	100	100
D-10	F	W	0	LD,AH	30	90	100	100
D-11	F	W	0	R-2	20	90	90	90
D-12	F	W	0		30	80	90	100
D-13	M	W	0	AH	20	80	60	90
D-14	M	W	0	R-2	20	100	80	100
D-15	F	W	0	LD,R-2	20	100	90	100
D-16	M	ME	0	LD,R-2	20	100	90	90
D-17	F	W	0		20	90	100	100
D-18	M	H	0	R-2	20	80	70	80
D-19	M	W	0		30	100	100	100
D-20	M	W	0		30	100	100	100
D-21	M	H	0	R-2	30	100	100	100
D-22	M	ME	0	R-2	20	100	100	100
D-23	F	B	0		30	80	60	100
D-24	M	H	0	R-2	30	100	100	100
D-25	M	B	1	R-2	30	100	100	70

KEY

Ethnicity
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- B - Black
- H - Hispanic
- ME - Middle Eastern

* Student had previous instruction with HANDS-ON EQUATIONS

Special Learner Codes
- AH - Attention Deficit Hyperactivity Disorder
- LD - Learning Different
- SE - Special Education
- R - At-Risk
  - R-2 Failed one or more portions of most recent TAAS
  - R-4 Excessive Absences
  - R-6 Retained in a previous grade

TABLE 1G/T: TEST RESULTS FOR CLASS G/T

Student	Sex	Ethn	Days Absent	Special Learner Codes	Pre	Post	3wk	6wk
G-1	M	W	0		40	100	100	100
G-2	F	W	0		20	100	100	100
G-3	F	H	1		50	90	100	100
G-4	M	W	0		60	100	100	90
G-5*	M	W	0		100	100	90	100
G-6	F	W	0		40	90	100	90
G-7	F	W	0		20	100	90	100
G-8	M	W	0		40	100	100	100
G-9	M	W	1		20	100	80	70
G-10	M	W	0		30	100	100	80
G-11	M	W	0		20	80	100	100
G-12	F	W	1		40	100	80	100
G-13	M	W	2		50	100	100	90
G-14	M	W	0	R-2	30	100	100	90
G-15	F	H	0		30	90	90	100
G-16	M	W	0		40	100	100	90
G-17	M	W	0		40	100	90	100
G-18	F	W	0		20	100	100	100
G-19	F	W	0		70	100	100	100
G-20	M	W	0		20	100	100	100
G-21	M	W	1		40	80	100	100
G-22	M	W	0		30	100	100	100
G-23	M	W	0		30	100	100	100
G-24	M	W	0		30	80	100	100
G-25	F	W	0		20	100	100	100
G-26	F	W	0		20	100	90	100
G-27	M	W	0		30	100	100	100
G-28	F	W	0		30	90	100	80
G-29	F	W	0		20	100	90	90
G-30	F	W	0		40	100	100	90
G-31	F	W	1		30	100	100	100

(Key given on next page)

(TABLE 1 G/T continued)

KEY	
Ethnicity W - White B - Black H - Hispanic ME - Middle Eastern * Student had previous instruction with HANDS-ON EQUATIONS	Special Learner Codes AH - Attention Deficit Hyperactivity Disorder LD - Learning Different SE - Special Education R - At-Risk R-2 Failed one or more portions of most recent TAAS R-4 Excessive Absences R-6 Retained in a previous grade

Class Mean Scores

The mean score for each class has been tabulated for each test. The mean was obtained by totaling all the scores from a particular test for a particular class and dividing the sum by the total number of students who were present that day and had completed the test from that class. The quotient is the class mean score.

TABLE 2A: CLASS MEAN SCORES ON ALL TESTS

Class	Pretest	Posttest	3 Week Retention	6 Week Retention
A	28.9	95.6	87.8	98.3
B	26.8	93.6	92.0	94.4
C	23.6	95.4	93.6	92.1
D	24.8	95.2	88.8	95.6
G/T	35.5	96.8	96.8	95.5

Student A-11's score decreased significantly from the posttest to the three-week retention test. Possible causes are discussed in Chapter V. The class mean score for Class A would be 91.8 if his score were deleted.

Other students also showed a 30-point decrease/increase or more from the posttest to the 3-week retention test or from the 3-week to the 6-week retention test. These students were A-7, B-5, B-13, B-20, C-5, C-7, C-11, C-19, D-13, D-23, and D-25. This extreme score change was not observed in class G/T. If these students' scores were deleted and the class mean scores recalculated, Table 2A would be replaced by Table 2B.

TABLE 2B: ADJUSTED CLASS MEAN SCORES

Class	Pretest	Posttest	3 Week Retention	6 Week Retention
A	27.3	96.3	93.1	95.1
B	27.3	93.6	95.0	95.0
C	23.9	95.0	96.1	95.0
D	24.5	96.4	90.9	96.3
G/T	35.5	96.8	96.8	95.5

Effects of Previous Instruction

Students had received instruction in linear equations four months prior to the administration of the pretest. This included problems of the form $x + a = b$, where a and b are integers. Solving for a missing addend has been taught since the first grade. Items #1 and #2 were of this form. The variable in item #3 had a coefficient other than one, which made this problem more difficult since it involved both addition and multiplication.

TABLE 3: PERCENT OF STUDENTS WITH CORRECT RESPONSES ON PRETEST ITEMS

Class	Correct Responses on Items #1 and #2	Correct Responses on Items #1, #2, and #3
A	82.3%	52.9%
B	84%	40%
C	86.3%	22.7%
D	96%	43%
G/T	100%	61.2%

Mastery Levels

The percent of students demonstrating 80% or better on each test for each class has been tabulated. The percent showing mastery is obtained by counting the total number of students who scored 80, 90, or 100% on the particular test in a class and dividing by the total number of students in that class that completed that test.

TABLE 4A: PERCENT OF STUDENTS DEMONSTRATING MASTERY
(Mastery Level = 80% or More on Test)

Class	Pretest	Posttest	3 Week Retention	6 Week Retention
A	0%	100%	88.9%	100%
B	0%	96%	88%	100%
C	0%	100%	90.9%	91.7%
D	0%	100%	84%	96%
G/T	3.2%	100%	100%	96.3%

If student A-11's deviant score were deleted, Class A would have had 94.1% of its students demonstrating mastery on the three-week retention test. If other unusual scores were also deleted as they were for the adjusted Table 2B (students A-7, B-5, B-13, B-20, C-5, C-7, C-11, C-19, D-13, D-23, and D-25), then the percent of students in each class showing mastery would be as shown in Table 4B below.

TABLE 4B: ADJUSTED PERCENTS FOR MASTERY
(Mastery Level = 80% or More on Test)

Class	Pretest	Posttest	3 Week Retention	6 Week Retention
A	0%	100%	100%	100%
B	0%	95.5%	100%	100%
C	0%	100%	100%	100%
D	0%	100%	90.9%	100%
G/T	3.2%	100%	100%	96.8%

CHAPTER V

INTERPRETATION OF DATA

The data gathered may be interpreted by examining the performance of the group as a whole, by separate class performance, and by individual student performance on the four tests. Observations from the four tests may aid in understanding reasons for increases and decreases in scores as well as deviant scores. From the completed study come ideas and recommendations for other studies of a similar nature as well as conclusions drawn from the data and observations.

Pretest

Results on the pretest were basically as expected with a few exceptions. Most students had correct responses for #1 and #2. (See Appendix A for pretest items.) Some students had #3 correct and a few had correct responses for #4. Most students had either 20% or 30% correct. The class mean scores show that in classes A, B, C, and D students averaged between two and three correct responses. These correct responses were almost exclusively items #1, #2, and #3 that had been taught previously in a symbolic manner. #1 and #2 are common forms seen as early as first grade, using a box instead of a letter variable for the missing addend. Because students had no experience with the more complex forms, correct responses on problems #4 - 10 were not expected. Class G/T averaged between three and four correct responses. Only one student out of all five classes demonstrated mastery on the pretest.

After students were first shown the manipulatives, each class was asked if anyone had ever used the system before. Also, each student who scored 50% or better on the pretest was asked individually if he/she had used the system . Student A-11 had used the system the previous summer at Scottish Rite Hospital: he scored 50%. Student B-24 indicated that he had used the system previously in fifth grade in another school district: he scored 50%. Student G-5 had used the system two years earlier in another state: he scored 100%. Student G-19 scored 70% and students G-4, B-2, and A-8 scored 60% on the pretest: these students had never used the system.

Posttest

The results of the posttest, which came immediately after instruction, were extremely good. All students demonstrated at least 80% mastery except student B-10. This student had missed two days of instruction and scored 0% on the pretest and 70% on the posttest. He came in the following week after the posttest during tutoring and made up the work he had missed. The class mean scores for the posttest show that in each class, the students averaged between nine and ten correct responses. High scores immediately following instruction were expected.

3-Week Retention Test

The results of the three-week retention test were somewhat lower. Only class G/T maintained 100% of the students demonstrating mastery by their scoring 80% or above. The mean score for class G/T was the same for the three-week retention test as it was on the posttest. All other class mean scores decreased at least 1.6 percentage points and as much as 7.8 percentage points from the posttest to the three-week retention test. This decrease in scores was anticipated due to the fact that there was no review of any type occurring between the posttest and the three-week retention test.

Student A-11 had a dramatic decrease in his three-week retention score. His pretest score was 50%; his posttest score was 100%; and his three-week retention test score was 20%. Observation of student A-11 during the time between the posttest and three-week retention test revealed poor effort and a general lack of a serious approach toward his academic responsibilities. Following the three-week retention test, the student was confronted by the teacher about his poor performance and encouraged to improve his efforts. The score of 20% is so deviant, it is probably not valid. The mean score for class A on the three-week retention test without student A-11 would be 91.8%. The percent of students demonstrating mastery on the three-week retention test would be 94.1% if A-11's score were deleted.

6-Week Retention Test

After six weeks of non-algebraic instruction and no review, the scores were expected to decrease even more. The increase in class mean scores for three out of five classes and the increase in the percent of students demonstrating mastery for four out of five classes was then a bit surprising. The mean scores for classes C and G/T decreased 1.5 and 1.3 percentage points respectively. (See Table 2A in Chapter IV.) Mean scores of classes A, B, and D increased 10.2, 2.2, and 6.8 percentage points, respectively. The same classes A, B, and D showed significant gains in the percent of students who demonstrated mastery. (See Table 4A in Chapter IV.) Only class G/T showed a decrease in mastery with one student, G-9, scoring 70%.

Individual Test Results

Many students' scores increased from the three-week retention test to the six-week retention test. This occurred after a decrease in many scores from the posttest to the three-week retention test. As a whole, many students entered into the three-week retention test with an overconfident, almost careless, attitude. Students were encouraged to check their answers, but many were satisfied with the first solution obtained and did not bother to check their solutions. Many of these students were genuinely surprised at their decrease in score from the posttest to the three-week retention test. Many were eager for another opportunity to prove their understanding of the concept. This opportunity came in the form of the six-week retention test, and students generally gave more concentrated effort to that test than to the three-week retention test. Checking the solution with the original equation was highly emphasized with the reminder that, as on the other tests, the students were not limited by time. Many more students, consequently, utilized the check method for every problem on the six-week retention test.

Student A-7's score increased from 70 on the three-week retention test to 100 on the six-week retention test. Student A-7 was a special education student who struggled consistently with most of the mathematical concepts all year. She was very excited about using the HANDS-ON EQUATION System because she experienced success and understanding while using the manipulatives. She seemed very disappointed with the score of 70 on the three-week retention test perhaps because that score is frequently labeled as a "poor passing grade." She gave a more concentrated effort on the six-week retention test and made use of the check on every problem. Her score increased thirty points.

Student B-5's score increased from 70 on the three-week retention test to 100 on the six-week retention test. Student B-5 was classified as at-risk because of excessive unexcused absences. The student is frequently inconsistent in effort, attitude, and performance. He was a bit surprised at his score of 70 on the three-week retention test which may have motivated him to check his answers more carefully on the six-week retention test. He improved 30 points. Student B-13's score decreased 30 points from the posttest to the three-week retention test. This student is also frequently inconsistent in performance. Student B-20's score also decreased 30 points from the posttest to the three-week retention test. He is usually a more consistent student. There is no documented accounting for this decrease in score.

Student C-5's score increased from 60 on the three-week retention test to 80 on the six-week retention test. This student was classified as learning different and had mathematics class the last period of the day. These two factors together may have been contributors to his decrease and increase in score. The student generally had to make a conscious effort to focus his attention. At the end of the day, he was frequently mentally exhausted.

Having made 100 on his posttest assured him of his capability and perhaps motivated him to improve his three-week retention score of 60. He improved 20 points. Student C-7's score decreased 30 points from the three-week retention test to the six-week retention test. This student is also classified as learning different. His decrease in score may also be attributed to mental exhaustion. Student C-11's score also decreased 30 points from the three-week retention test to the six-week retention test. Student C-19's score decreased 30 points from the posttest to the three-week retention test. There is no documented accounting for the decrease in score for students C-11 and C-19.

Student D-13's score increased from 60 on the three-week retention test to 90 on the six-week retention test. Student D-13 is classified as having attention deficit hyperactivity disorder. This student had not focused his energies in the academic direction during the grading period that contained the pretest, posttest, and three-week retention test. He received a failing grade in mathematics on his report card two days prior to the six-week retention test. This may have motivated him to take a more serious and focused approach to the six-week retention test. His score on the six-week retention test was 30 points higher than the three-week retention test and 10 points higher than the posttest.

Student D-23's score was 80 on the posttest, 60 on the three-week retention test, and 100 on the six-week retention test. This was an honor student who was a very consistent worker. She was surprised by the low score on the three-week retention test. She used the check method to confirm her solutions on the six-week retention test. Her score increased 40 points.

Student D-25's score decreased 30 points from the three-week retention test to the six-week retention test. This student is frequently inconsistent in performance. He also has a tendency to become overconfident. His overconfidence may have influenced him to disregard the check method.

As with many aspects in memory, perhaps these students needed incubation time for the concepts to "take root" or germinate. From the data, one might conclude that the time lapse from the posttest to the six-week retention test was not detrimental to the retention of the concept. Student C-9 and C-21 were absent for the three-week retention test and yet at the six-week retention test, C-9 increased his score from 90 on the posttest to 100 on the six-week retention test with no review. He has attention deficit hyperactivity disorder. Student C-21, a learning different and at-risk student, maintained her score of 100 on the posttest to the six-week retention test.

Use of Manipulatives

Manipulatives were not used on the pretest since students were not familiar with the HANDS-ON EQUATIONS materials at that time. Manipulatives were distributed to all five classes for the other three tests. On the posttest, three-week retention test, and six-week retention test, many students improvised with the manipulatives. This was necessary because some of the test items called for the use of more number cubes than each desktop set of manipulatives provided. Students had minimal experience with the pictorial representation, so many students were reluctant to use it on the tests. Some simply discovered the distributive property and multiplied the constant inside the parentheses by the number outside. Others tore out small pieces of paper and either put "x" on it or a number value, depending on their needs.

Because the posttest immediately followed instruction on using the pictorial method, several students chose to use only the pictorial method on the posttest. Most of these students were in class G/T and did not set up the manipulatives at all. Many other students used the manipulatives on all problems, including items #1 and #2, which they had completed successfully on the pretest without manipulatives. But by the three-week retention test, all students were using the manipulatives for at least a portion of the test. Some students would begin with one method and, if they experienced difficulty, would change to the other method to help clarify the solution. Students were observed changing from the pictorial representation to the manipulatives as well as from the manipulatives to the pictorial representation. The majority of the students used the pictorial method for at least one problem on each test. Most students used the "triangle" or pawn notation, some used X's, and a few used tally marks in their pictorial notation. After solving the equations using manipulatives, some of the G/T students were observed checking their solutions symbolically, substituting in the value for the variable in the original equation in its symbolic form.

Recommendations for Future Studies

In this study, students worked with manipulatives for five days, and only part of the fifth day was spent on the pictorial model. If a similar study were to be conducted, it would be advisable to allow one complete teaching day per worksheet with additional time for practice with the pictorial method. At the completion of worksheet #7, students should perhaps rework previous worksheets using only the pictorial method or a combination of pictorial with manipulative check. This would give students more practice and confidence before encountering problems on a test that required the pictorial method because of the limited nature of the manipulatives.

Another study that might prove to be of interest and worth would involve three classes of comparable ability. One class would be taught with manipulatives, another class taught exclusively with the pictorial method, and the third class taught using only the traditional symbolic approach. Each class would then be tested, not only for mastery but also for retention.

Conclusion

"A learning occurrence takes place when the stimulus situation together with the contents of memory affect the learner in such a way that his or her performance changes from a time before being in that situation to a time after being in it. The change in performance is what leads to the conclusion that learning has occurred." (Gagné, 4) Given the pretest data and the posttest, three-week retention test, and the six-week retention test data, it can be concluded that each student learned. 100% of the students demonstrated at least 80% mastery on at least two out of the three tests that followed the instruction. 87% of the students demonstrated mastery on all three tests that followed instruction. Students in the regular education classes performed comparably to the students in the gifted/talented class. Students that qualify for special education assistance and at-risk remedial instruction, as well as the learning different students and attention deficit hyperactive students, performed as well as those in regular education without those classifications. Their performance was also comparable to that of the gifted/talented students. This is not unexpected, given that these special learners are disproportionately experiential, practical, hands-on learners. That is, they learn best through methods that require the use of multiple senses, particularly the tactile sense. (Sagor, 32)

The increase in scores from pretest to posttest is notable but even more worthy of note is the relative stability and maintenance of mastery level scores. The strength of the study is not simply that the students learned with the manipulatives, but that they were able to retain the concept without further instruction or review, given only the prompt of the manipulative materials during the test administration.

Manipulatives in general allow teachers to create situations that draw mathematical responses from students. Students are actively involved in the creation of mathematics, resulting in improvements in motivation, involvement, understanding, and achievement -- overwhelming reasons to believe that manipulatives produce good mathematics. (Herbet, 4)

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APPENDICES

APPENDIX A

Pretest, Posttest, Retention Tests (3-Week, 6-Week)

PRETEST

Find the value of x for each:

1) $x + 6 = 9$ $x =$

2) $10 = 8 + x$ $x =$

3) $2x + 4 = 10$ $x =$

4) $3x = x + 2$ $x =$

5) $13 + x + x = x + 3 + 3x$ $x =$

6) $x + 3x + 12 = 2x + 20$ $x =$

7) $6 + 2x + 4 - x = 2x + 3$ $x =$

8) $2(x + 5) = 4x + 2$ $x =$

9) $2(3x + 1) = 3x + 20$ $x =$

10) $3(x + 1) = 10 + 2x$ $x =$

POSTTEST

Find the value of X for each:

1) $x + 7 = 11$ $\quad x =$

2) $9 = 4 + x$ $\quad x =$

3) $3x + 2 = 11$ $\quad x =$

4) $4x = x + 6$ $\quad x =$

5) $x + 8 + x = 3x + 2 + x$ $\quad x =$

6) $2x + x + 16 = x + 18$ $\quad x =$

7) $5 + 3x + 3 - x = 5x + 2$ $\quad x =$

8) $2(x + 8) = 5x + 1$ $\quad x =$

9) $2(2x + 2) = 3x + 10$ $\quad x =$

10) $3(2x + 1) = 13 + x$ $\quad x =$

THREE-WEEK RETENTION TEST

Find the value of X for each:

1) $x + 8 = 12$ x =

2) $7 = 4 + x$ x =

3) $2x + 6 = 14$ x =

4) $3x = x + 10$ x =

5) $x + x + 12 = x + 5x + 8$ x =

6) $5 + 3x + x = 3x + 9$ x =

7) $3 + 4x - x + 11 = 5x + 6$ x =

8) $3(x + 2) = 4 + 4x$ x =

9) $2(2x + 5) = 3x + 13$ x =

10) $3(x + 1) = 5 + x$ x =

SIX-WEEK RETENTION TEST

Find the value of X for each:

1) $x + 9 = 13$ x =

2) $8 = 6 + x$ x =

3) $3x + 5 = 20$ x =

4) $4x = x + 9$ x =

5) $11 + x + x = x + 5 + 4x$ x =

6) $6 + 2x + 3x = x + 18$ x =

7) $4 + 2x + 16 - x = 3x + 4$ x =

8) $3(x + 3) = 4x + 8$ x =

9) $2(3x + 2) = 20 + 2x$ x =

10) $2(2x + 3) = 2x + 16$ x =

APPENDIX B

Teacher's Daily Journal Entries

MAR-17-2009 10:28AM FROM-BORENSON & ASSOC. 610-398-7863 T-110 P.008/014 F-364

TEACHER'S DAILY JOURNAL ENTRIES

Observations from Pretest (See Appendix A for copy of Pretest)

Most students got items #1, 2, and 3 correct. Many had correct answers also for #4. A few had correct answers for #5. From their responses, I gathered that many students thought x varied even within a problem or that a coefficient is added to the variable

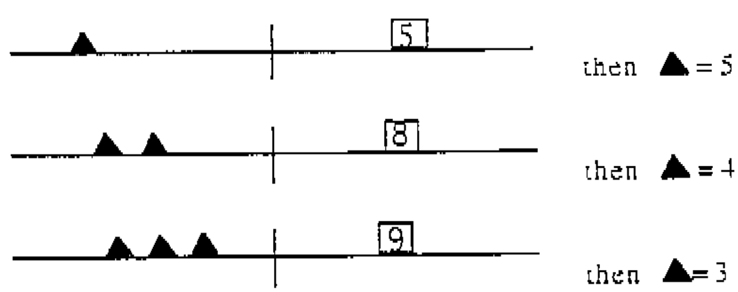
Examples: $3x = x + 2$, then, $x = 0$ and 1
$3+0 = 1 + 2$,

I did not discourage guessing an answer, since guess and check is a legitimate problem solving method.

Observations from Daily Lessons (See HANDS-ON EQUATIONS, Level I, in Appendix C)

Day 1: I introduced lesson #1 with the balance/scale model and asked "What is this?" referring to the balance beam. We discussed the idea of a balance or scale. I asked "If the scale is balanced and then I put something on the left side, what must I do to maintain balance?" Students responded, "Put the same thing on the right side".

Example:



MAR-17-2009 10:29AM FROM-BORENSON & ASSOC. 610-398-7863 T-110 P.009/014 F-364

Students solved these orally with no difficulty. I introduced the name for the blue pawn as "x" and then proceeded by passing out materials. I allowed a few minutes for exploration and then distributed worksheets and continued with the lesson. The worksheet showed a pictorial representation that the students used to set up their manipulatives. We did the first few problems together, and then I allowed students to work ahead if they wished. This caused problems because, when students finished, they did not check to see if their answers were correct; they proceeded to play and build with the game pieces. Students who didn't work ahead became distracted by the others and didn't work well (or at all in some cases).

By the afternoon, I modified instruction by not allowing students to work ahead. This made it much easier to check each student's set-up of the equation and pinpoint errors more quickly. We achieved more in the afternoon classes than in the morning by doing this. The basic problem solving method used during the lesson was trial and error, and checking one's work became mandatory.

Day 2: I began class with a review and then made a leap from pictorial notation to symbolic notation for the set-up of the manipulatives. I reminded students that another name for the blue pawn is "x". I asked students how we would set up $2x + x = x + 8$. Some attempted

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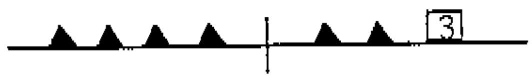
using a number cube as the coefficient. But other students were quick to correct this, stating that 2 ▲ meant 2 + x, not 2x. So we agreed upon

for the set-up. I distributed materials and worksheet #2. We worked each problem after students solved it at their desks. (Students were not allowed to work ahead.)

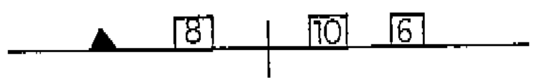
Difficult problems to set up included #3: $x + 4 = 2x + 3$. Students read $x + 4$ the same as $4x$.

Improper setup



Correct setup

Review problem #7



(shown pictorially on the worksheet) was also difficult at first. This caused some confusion because students had only two number cubes with values 5 and up. At first they said, "We can't do this one". I said, "Be creative", so then they adapted with various correct responses.

All of the morning classes finished in time for me to introduce the problem: $4x + 2 = 3x + 9$. I let them use the only method we had discussed so far -- guess and check. After a length of time, some students had an answer, but the majority were either still working or had given up. I then introduced the concept of a "legal move", in this case the removal of the same number of pawns from both sides of the equation.

Day 3: I reviewed at the beginning of the class the new method or "legal move" used with $4x + 2 = 3x + 9$. Then I distributed the manipulatives and set up another problem: $5x + 2 = 2x + 14$. We worked it together using the legal move of removing pawns from both sides of the equation.

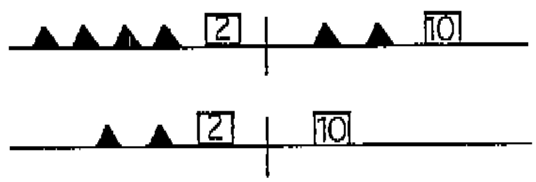
Class G/T (See Table G/T in Chapter IV) immediately noticed the other legal move which involves the number cubes. Class A also noticed the move, but was not convinced it would always work until they had several successes with it.

All classes finished worksheet #3.

I introduced the legal move with the number cube. We all started worksheet #4, but no class finished.

Day 4: We finished worksheet #4 in class. All students recognized the legal moves with the pawn, but a few students were reluctant to use the legal move with the number cubes. Many would rather do legal moves with pawns and then solve by guessing. [Perhaps if the student could manipulate individual units representing the constant, they would be more comfortable with this legal move.]

Example: $2x + x + x + 2 = 2x + 10$



Then they figured, "2 plus what equals 10? 2 + 8, so each pawn = 4." I encouraged the reluctant to at least try and practice the number cube legal move, but I told them that after they had learned to use it, if they still didn't like it, they didn't have to use it.

I introduced subtraction by writing $5x - 3x + 2 = x + 5$ and asking how to set up the problem. The students said "put out five pawns, then take away three pawns...." Very few tried to "add on" rather than "take away". We all started worksheet #5.

Day 5: We reviewed subtraction. I passed out manipulatives. We completed worksheet #5

I introduced lesson #6. I explained that we would not be doing any new legal moves. The lesson was just understanding a new notation. I wrote the problem $2(x + 3) = x + 8$. Some students looked a little worried. I reminded them

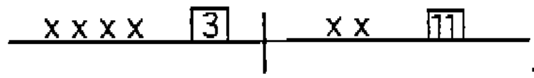
"On a problem like this, what do we do first?" The students responded "Whatever is in the parentheses." I affirmed this and remarked that the number next to the parentheses just told how many times to do what was in the parentheses. The set up for $2(x + 3) = x + 8$ is thus:

Then I asked, "So where is the number 2 in the set up?" They responded that it was in the fact that they set up x + 3 twice. We began worksheet #6 - all classes except Class D finished Worksheet #6. I introduced (to all but Class D) the pictorial method.

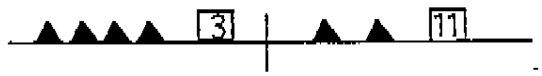
Day 6: Class D was interrupted to go to a band concert. By the time we got back to class, there was only about 15 minutes left. I introduced the pictorial method and we practiced it. We agreed to take the posttest during silent readingtime (after lunch). This allowed class D plenty of time to complete the test and an equal amount of time for instruction. All other classes reviewed the pictorial method for 15 to 20 minutes, then I gave the posttest. I encouraged students to do the check on each problem but did not require it. Most students completed the test in 10-15 minutes. Some took 20-25 minutes to complete it. Two problems required the use of the pictorial method because of the limitation of the number cubes.

MAR-17-2009 10:29AM FROM-BORENSON & ASSOC. 610-398-7863 T-110 P.014/014 F-364

Regarding pictorial notation, many students preferred the quicker notation for $4x + 3 = 2x + 11$, that is, using X's on the balance scale diagram:



But equally as many (if not more) preferred the notation that resembled the actual pawns:



Also, some students set up 11 as $10 + 1$ because the number cubes only numbered to ten. I pointed out that the nice thing about the pictorial notation is that we weren't limited with our numbers.

APPENDIX C

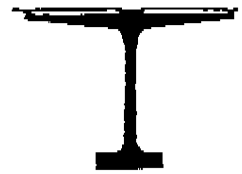
Lessons From HANDS-ON EQUATIONS, Level I and Teacher's Notes

MAR-17-2009 11:35AM FROM-BORENSON & ASSOC. 610-398-7863 T-111 P.002/008 F-368

Materials Needed by the Teacher:

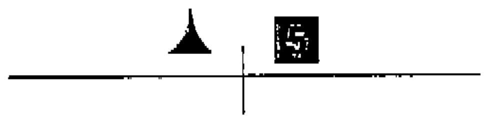
Pawns and cubes as above, but larger:

A stationary physical scale

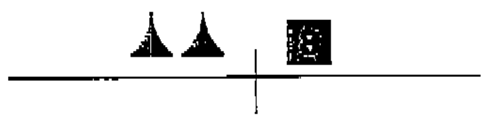


Lesson #1

In the first lesson, the teacher displays on the physical scale in front of the classroom, problems such as

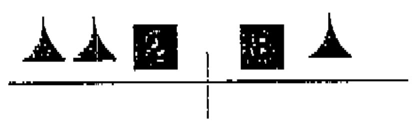


and



FOR HOME USE:
Please display these problems on the enclosed laminated balance scale.

Once students grasp the concept that both sides of the scale must have the same value for the scale to balance, they see that the pawn in the first problem is worth 5, and that in the second problem is worth 4. Students can then be presented with other "physical equations" which they are to solve by trial and error methods.

Ex. 

The students see that "1" does not work since both sides are not equal. "2" does not work, etc. "6" does work since the left side is now 14 and so is the right side. The students are informed that the pawn has a special name. "x," and that there is a special way of writing the answer:

$$x = 6, \text{ check: } 14 \stackrel{\checkmark}{=} 14.$$

The students are given Student Kits so that they can set up the worksheet problems at their desks. (On the student setup, it is helpful if the number cubes are facing upward so that the teacher can easily check if the students have the correct setup.)

Comments on Lesson #1

In this lesson, students begin learning about equations, variables, and unknowns on both sides of a setup—but they do so intuitively, through Piagetian learning.[3] Indeed, the word "variable," which can even scare some adults, is not used at all. Important algebraic concepts are nonetheless acquired in a very natural way as the students work with the materials.

Lesson #2

Students are reminded that the pawn has a special name, "x." Therefore, the problem

$$2x + x = x + 8$$

really calls for placing "two x's" and "one x" on the left side of the scale, and "an x" and an "8-cube" on the right side:

The students are given their Student Kits so that they can set up the problem at their desks. Then, they can solve by trial and error methods as in Lesson #1. The answer is:

$$x = 4, \text{ check: } 12 \stackrel{\checkmark}{=} 12.$$

Other examples which the teacher may use in this lesson include

$$3x + 1 = x + 7$$

and

$$4x = 3x + 5.$$

Comment on Lesson #2

It is a tremendous credit to the power of this system that young students can interpret and make sense of the above problems after only two lessons!

Lesson #3

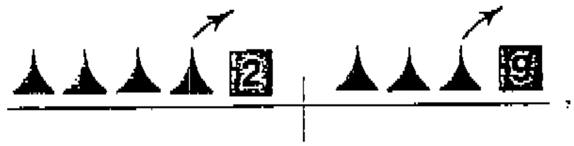
The teacher begins by posing to the class a problem such as

$$4x + 2 = 3x + 9.$$

The students set up this problem at their desks and attempt to solve it. Because this problem may stymie many students, it offers an excellent opportunity for the teacher to say:

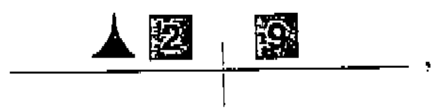
> "Would you like to learn an easier way of getting the answer than by using trial and error?"

The teacher can now proceed to see if the students "buy" the idea (which the teacher now physically demonstrates), that if one pawn is removed simultaneously from each side of the balanced setup,



that the scale will still balance. Such a move, which leaves a balanced scale in balance, is called a "legal move." (To confirm that the students do in fact understand this key concept, the teacher should then attempt to remove two pawns from the left side and one pawn from the right side. The students should see that this move is <u>not</u> legal.)

By carrying out the above legal move two more times, the setup now shows



from which the students can easily see that $x = 7$. The teacher can now say to the class:

> "Let's see if our method has worked. Let's physically set up the original problem one more time to see if $x = 7$ makes both sides balance."

After setting up the initial problem, a student can come up to the front of the room to verify that if each pawn is 7, the system

balances since $30 \stackrel{?}{=} 30$. So the answer $x = 7$ is correct.

MAR-17-2009 11:35AM FROM-BORENSON & ASSOC. 610-398-7863 T-111 P.005/008 F-368

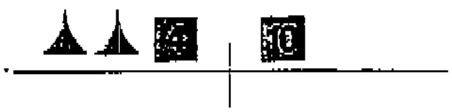
The teacher can now assign other similar problems for students to set up and solve at their seats, using legal moves if they wish:

$$\text{Ex. } 5x + 2 = 2x + 14$$

$$\text{Ex. } 2x + x + 4 = 4x + 1$$

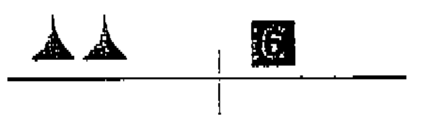
Lesson #4

In this lesson, students learn that subtracting the same number-cube value from each side of a balanced setup leaves the setup in balance.

Ex. 

Given the above setup, students can subtract a 4-value from the cubes on each side,

thus leaving

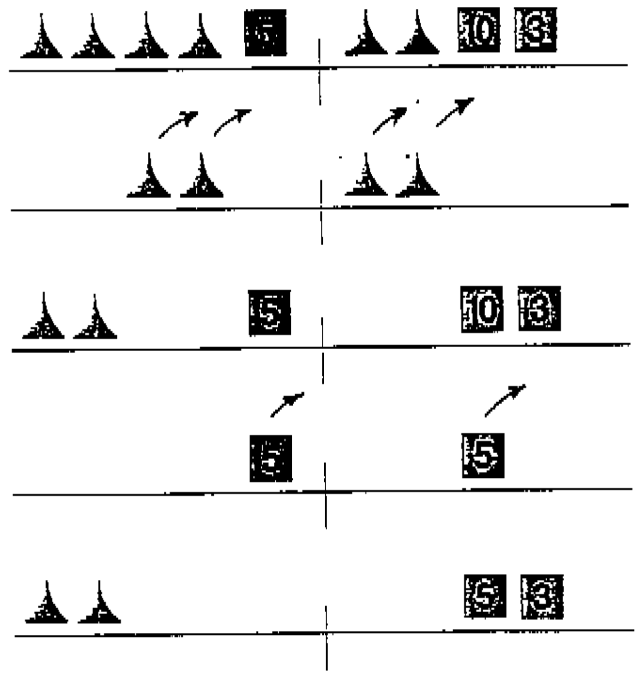


Sometimes this process if more clearly illustrated if the 10-cube is first replaced by a 4-cube and a 6-cube.

before removing the 4-value from each side.

So far, then, the students have learned two legal moves: that they may substract the same number of pawns from each side of a balanced setup and that they may subtract the same number-cube value from each side of the setup. The following example allows students to practice both legal moves. A possible solution is shown.

Ex. $4x + 5 = 2x + 13$



So, $x = 4$. The check in the initial setup reveals that $21 \stackrel{\checkmark}{=} 21$.

Other problems which can be given in this lesson include

$$2x + x + x + 2 = 2x + 10$$

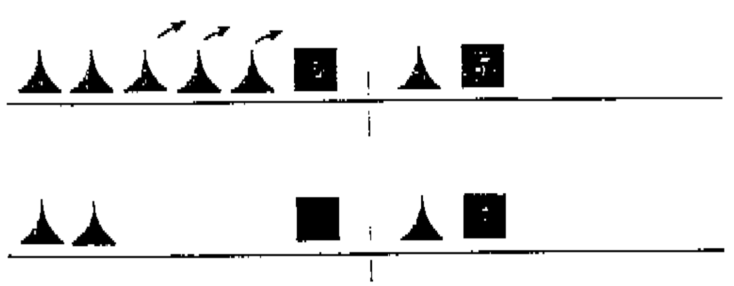
and

$$x + 3x + 3 = x + 18.$$

Lesson #5.

In this lesson, students take away pawns as part of the setup process:

$$5x - 3x + 2 = x + 5$$



From this, the initial setup, students can now proceed to use legal moves. After removing one pawn from each side, we see that $x = 3$. The check, in the above setup, shows that $8 \stackrel{\checkmark}{=} 8$.

Other examples which the teacher can assign in this lesson include

$$2x + x - x + 1 = x + 9$$

and

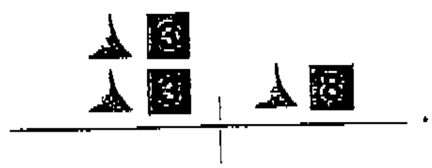
$$4 + 3x - 2x + x = x + 5.$$

Lesson #6

In this lesson, students learn to solve such equations as

$$2(x + 3) = x + 8.$$

They learn that the "2" outside the parenthesis means that what is inside the parenthesis, the "$x + 3$," is to be doubled. Hence the setup for this problem is



By having the students display the doubled portion in two rows, the teacher can easily check that the correct elements have been doubled. From here, the students can solve as in prior lessons. By subtracting one x from each side, we see that $x = 2$; check: $10 \stackrel{\checkmark}{=} 10$.

Other examples the teacher can give in this lesson include

$$2(2x + 1) = 18$$

and

$$2(x + 4) + x = x + 16.$$

Comment on Lesson #6

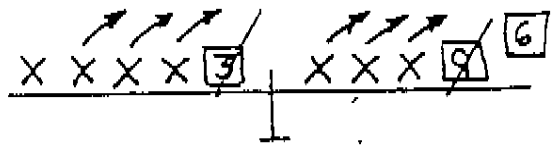
It is fascinating to see that elementary school students, when instructed in this manner, do not have any trouble in working with a multiple of a parenthetical expression. Often, they "discover" the distributive law on their own and double each element inside the parenthesis in sequence.

Lesson #7

In this lesson, students transfer their concrete, hands-on experience in solving algebraic linear equations to a pictorial system involving only pencil and paper. The technique is illustrated in the two examples below.

Ex. $4x + 3 = 3x + 9$

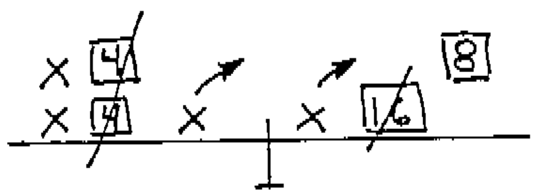
Solution:



So, $x = 6$. Check: $27 \stackrel{\checkmark}{=} 27$.

Ex. $2(x + 4) + x = x + 16$

Solution:



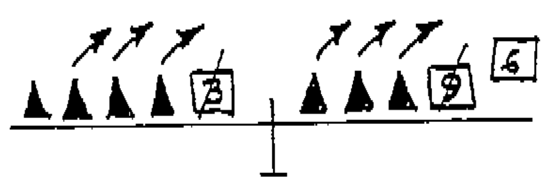
So, $x = 4$. Check: $20 \stackrel{\checkmark}{=} 20$.

Thus in this pictorial notation, the student draws a scale, and on it, places written "x's" (instead of pawns), places written boxed-numerals (instead of number cubes), and crosses off or places arrows above anything that is to be taken away. Once the answer is obtained, the student goes back to the initial pictorial setup *(redrawn, for clarity)* in order to carry out the check.

Note: Some students prefer to use pictures of the pawns, rather than written "x's," to represent the physical pawns.

Ex. $4x + 3 = 3x + 9$

Solution:



So, $x = 6$. Check: $27 \stackrel{\checkmark}{=} 27$.

This pictorial notation more closely resembles the actual physical setup and is therefore easier for some younger students and some students with learning disabilities. This notation is perfectly acceptable. (The picture of the blue pawn is shaded in, to distinguish it, in Level II, from the picture of the white pawn.)